Sparsity Methods for Systems and Control

Maximum Hands-off Control

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L^0 norm of a function

- The support of a function $u(t)$, $t \in [0,T]$:

$$\mathrm{supp}(u) \triangleq \{t \in [0,T] : u(t) \neq 0\}.$$

- The L^0 norm of a function $u(t)$:

$$\|u\|_0 \triangleq \mu(\mathrm{supp}(u)),$$

    - $\mu(S)$ is the Lebesgue measure (i.e. the length) of a subset $S \subset [0,T]$.

- L^0 norm: the total length of time durations on which the signal takes nonzero values.

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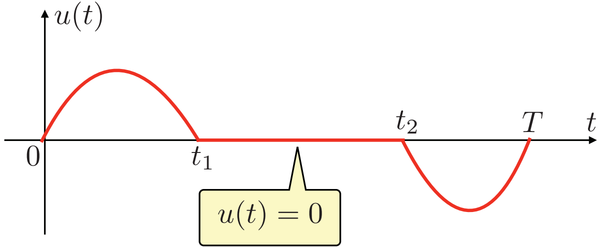
Example: L^0 norm of a function

- The L^0 norm

$$\|u\|_0 = \mu(\mathrm{supp}(u)) = t_1 + (T - t_2) = T - (t_2 - t_1).$$



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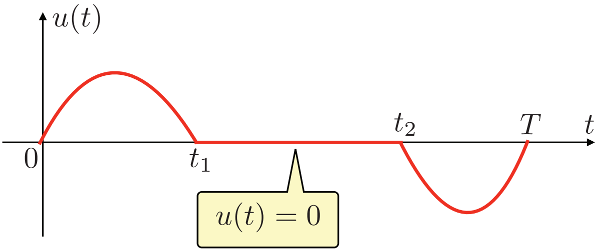

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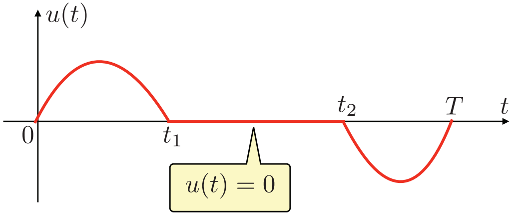
Practical benefits of sparsity in control

- Let us consider the sparse control signal $u(t)$, $t \in [0, T]$.
- Actuators as electric motors need energy to generate power.
- If the control $u(t)$ is sparse, we can stop energy supply to the actuator over the time interval $[t_1, t_2]$.
- Such a control is called a hands-off control.
  - This is also known as coasting.
- We can also reduce CO or CO2 emissions, noise, and vibrations.



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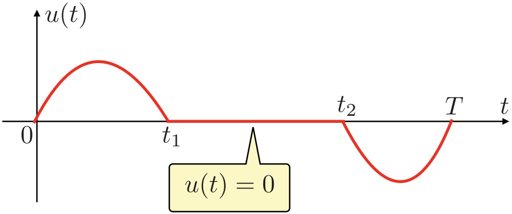

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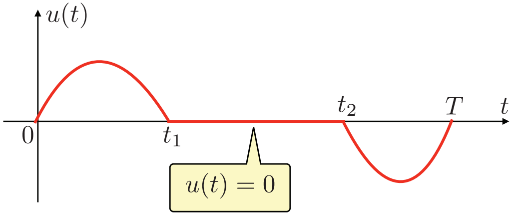

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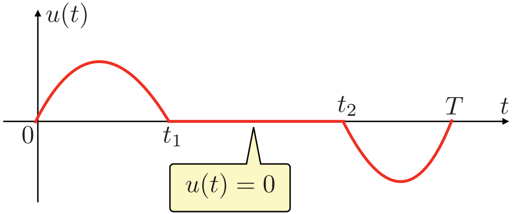

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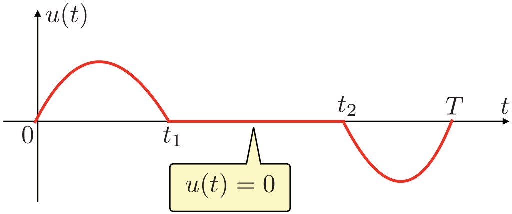

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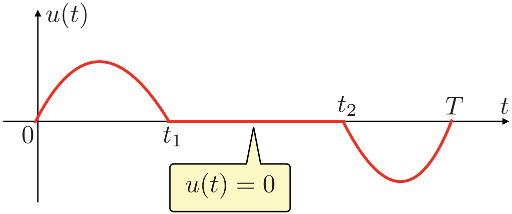

Examples of hands-off control

- Start-stop system in vehicles
- Hybrid cars
- Electric locomotives


1. https://www.carprousa.com/Understanding-Vehicle-StartStop-Systems/a/3
2. https://en.wikipedia.org/wiki/Hybrid_vehicle
3. https://en.wikipedia.org/wiki/Electric_locomotive

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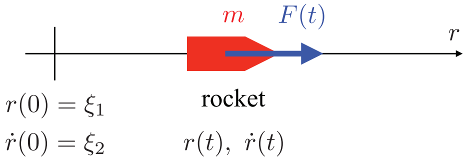
A simple example: rocket



- Control objective: Given $T > 0$, find $F(t)$, $0 \le t \le T$, such that

$$r(T) = 0, \quad \dot{r}(T) = 0.$$

- System model: $m\ddot{r}(t) = F(t)$ (Newton's second law of motion)
- State variable:

$$x(t) \triangleq \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} \Rightarrow \dot{x}(t) = \begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix} u(t).$$

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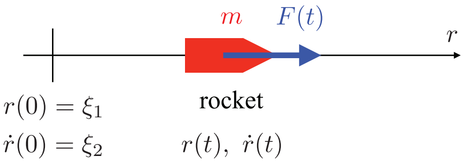

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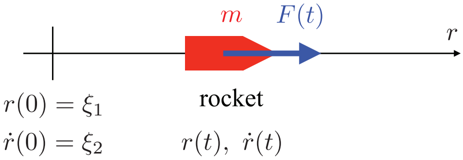

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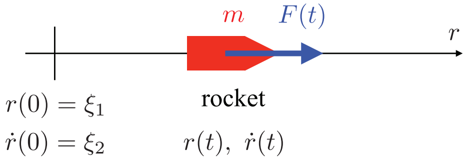

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Maximum hands-off control

Control system (rocket)

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Maximum hands-off control problem

Find a feasible control that minimizes the L^0 norm of u:

$$J_0(u) = \mu(\mathrm{supp}(u)) = \int_0^T |u(t)|^0 dt \quad \text{(the length of the support)}$$

L^0 norm and L^1 norm

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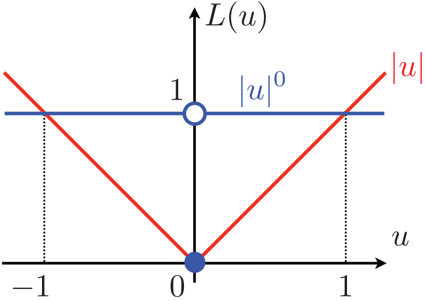

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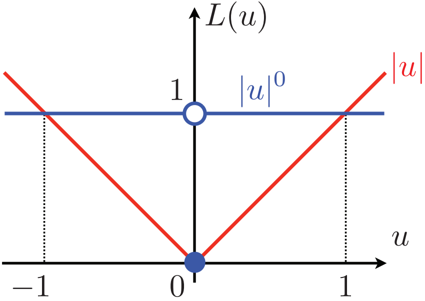

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$$J_1(u) = \int_0^T |u(t)| dt.$$

- Also known as fuel optimal control.
- A convex optimization problem.
- A closed-form solution is obtained.

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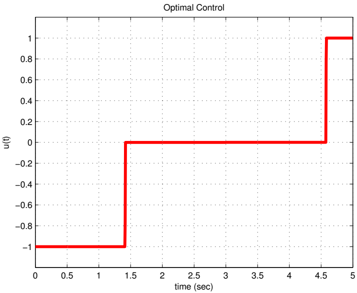
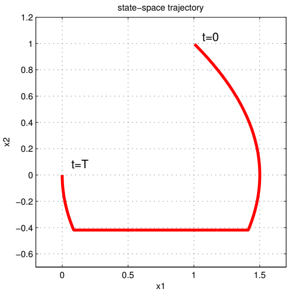
A simple example

L^1-optimal control $u^*(t)$ and trajectory $x^*(t)$ [Athans and Falb, 1966]





- $u^*(t) \equiv 0$ over $[3 - \sqrt{10}/2, 3 + \sqrt{10}/2] \approx [1.4, 4.6]$
- $u^*(t)$ is sparse ($\|u^*\|_0 = |\mathrm{supp}(u^*)| \approx 1.84 < 5 = T$)
- In fact, it is the sparsest (i.e., maximum hands-off control).

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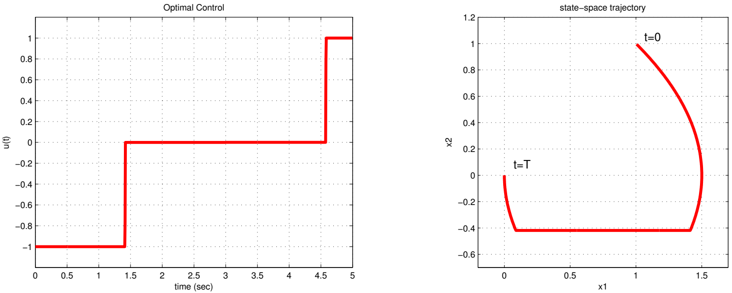

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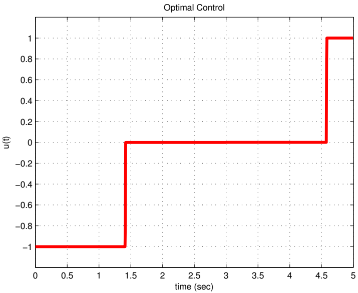

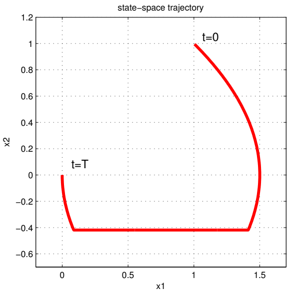

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For the linear time-invariant system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad t \geq 0, \quad \boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d,$$

find a control $\{u(t) : t \in [0, T]\}$ with $T > 0$ that minimizes

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subject to

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$$\|u\|_\infty \leq 1.$$

- This is difficult!

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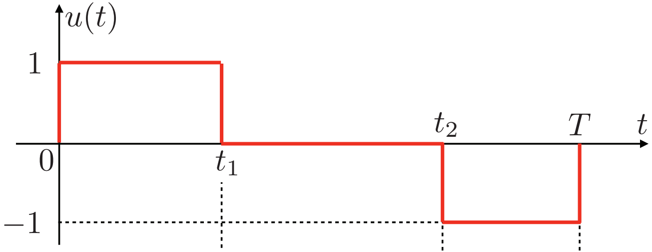
Bang-off-bang control

Theorem

If $(A, \boldsymbol{b})$ is controllable and A is nonsingular, then the L^1 optimal control $u(t)$ takes ± 1 or 0 for almost all $t \in [0, T]$ (if it exists).

- A control that takes ± 1 or 0 is called a bang-off-bang control.



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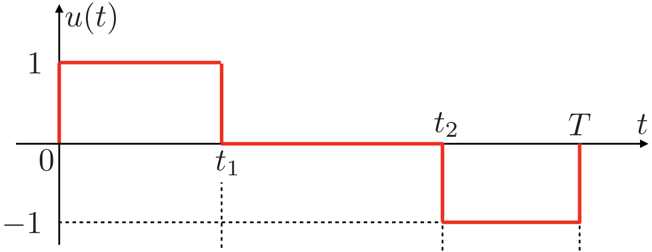


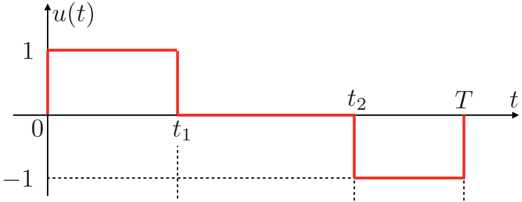
Equivalence between L^0 and L^1 optimal controls

Theorem

Assume that there exists an L^1-optimal control that is bang-off-bang. Then it is also L^0 optimal.

Theorem

Assume that there exists at least one L^1-optimal control. Assume also that $(A, \boldsymbol{b})$ is controllable and A is non-singular. Then there exists at least one L^0-optimal control, and the set of L^0-optimal controls is equivalent to the set of L^1-optimal controls.



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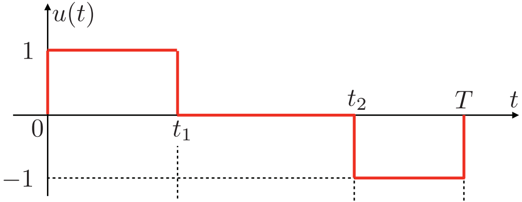
Assume that there exists an L^1-optimal control that is bang-off-bang. Then it is also L^0 optimal.

Theorem

Assume that there exists at least one L^1-optimal control. Assume also that $(A, \boldsymbol{b})$ is controllable and A is non-singular. Then there exists at least one L^0-optimal control, and the set of L^0-optimal controls is equivalent to the set of L^1-optimal controls.



Conclusion

- Maximum hands-off control is described as L^0-optimal control.
- Under the assumption of non-singularity, that is, $(A, \boldsymbol{b})$ is controllable and A is nonsingular, L^0-optimal control is equivalent to L^1-optimal control.
- Maximum hands-off control is a ternary signal that takes values of ± 1 and 0. Such a ternary control is called a bang-off-bang control.