Sparsity Methods for Systems and Control

Dynamical Systems and Optimal Control

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Table of Contents

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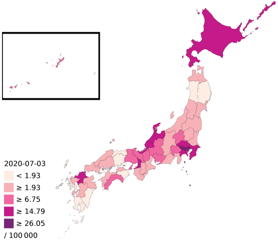
- A dynamical system is a moving system.
- industrial products
  - vehicles, airplanes, motors, electric circuits, etc.
- movement of planetary, change of weather, ant swarm, cell movement, fluctuations in stock prices, and spread of virus.
- In Part II, we will learn sparsity methods for dynamical systems.



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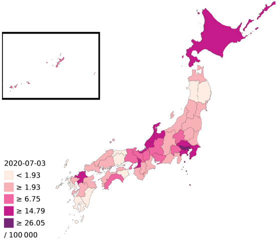

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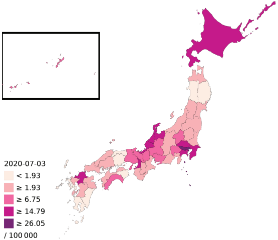

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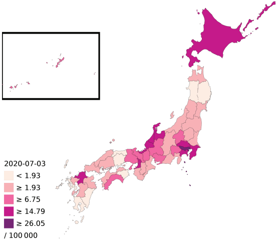

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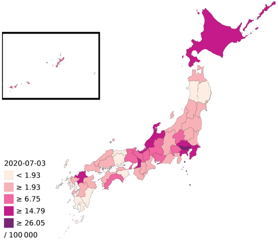

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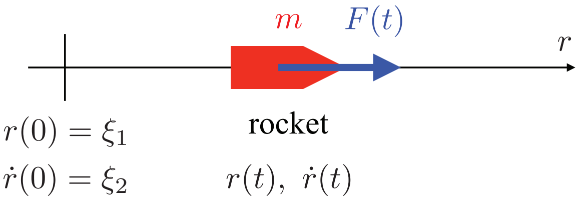
Rocket as a dynamical system

- A rocket in the outer space
  - No friction nor gravity acts
- The mass of the rocket is m [kg]
- The position and velocity are $r(t)$ [m] and $v(t) = \dot{r}(t)$ [m/s]
  - The initial condition: $r(0) = \xi_1$, $v(0) = \dot{r}(0) = \xi_2$
- The thrust force is $F(t)$



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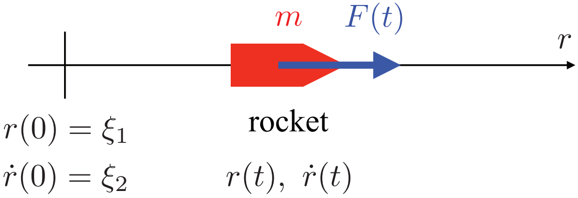

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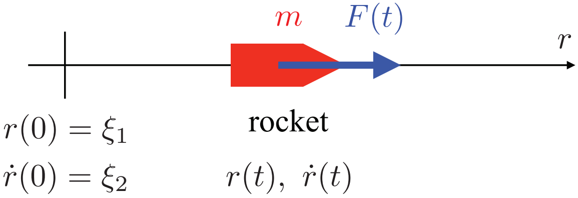

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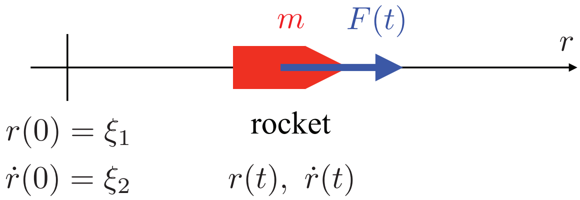

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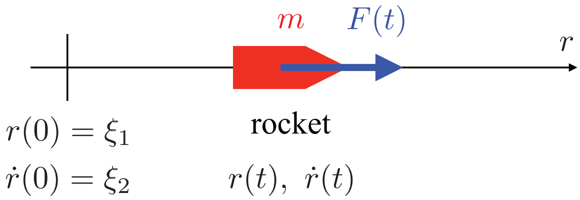
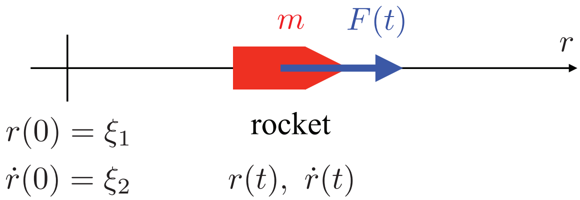


Differential equation



- The Newton's second law of motion gives

$$m\ddot{r}(t) = F(t), \quad r(0) = \xi_1, \quad \dot{r}(0) = \xi_2.$$

State

- The ordinal differential equation (ODE) of the rocket

$$m\ddot{r}(t) = F(t), \quad r(0) = \xi_1, \quad \dot{r}(0) = \xi_2.$$

- Define the state $x(t)$ by

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix}.$$

- Then we have

$$\dot{x}(t) = \begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ m^{-1}F(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\triangleq A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{=x(t)} + \underbrace{\begin{bmatrix} 0 \\ m^{-1} \end{bmatrix}}_{\triangleq b} \underbrace{F(t)}_{\triangleq u(t)}$$

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State equation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad t \geq 0, \quad \boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d,$$

- $\boldsymbol{x}(t)$: state
- $\xi = [\xi_1, \xi_2] = [r(0), \dot{r}(0)]^\top$: initial state
- $u(t)$: control

The solution

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-\tau)}\boldsymbol{b}u(\tau)d\tau, \quad t \geq 0.$$

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State transfer problem

- The initial state $x(0) = \xi$ is observed at time $t = 0$.
- Find a control $u(t)$, $0 \leq t \leq T$ that drives the state $x(t)$ from a given initial state ξ to the origin $\mathbf{0}$ in a given time $T > 0$.

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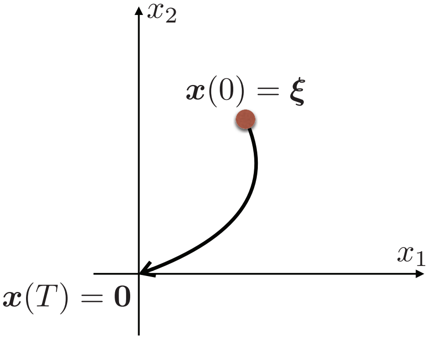

Controllability

- State equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad t \geq 0, \quad \boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d, \qquad (\star)$$

Controllability

We call the system $(\star)$ is controllable if for any initial state $\boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d$, there exist a time $T > 0$ and control $u(t)$, $0 \leq t \leq T$ such that the state $\boldsymbol{x}(t)$ in $(\star)$ is driven to the origin at time $t = T$, that is $\boldsymbol{x}(T) = \boldsymbol{0}$.

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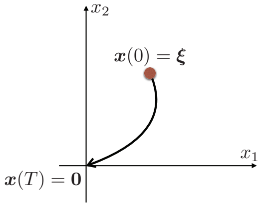

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Theorem

The dynamical system $(\star)$ is controllable if and only if the following matrix called the controllability matrix

$$M \triangleq \begin{bmatrix} \boldsymbol{b} & A\boldsymbol{b} & A^2\boldsymbol{b} & \dots & A^{d-1}\boldsymbol{b} \end{bmatrix}$$

is non-singular.

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- Suppose a dynamical system is controllable.
- Then for any $\xi \in \mathbb{R}^d$, any $\zeta \in \mathbb{R}^d$, and any $T > 0$, there exist a control $u(t), 0 \leq t \leq T$ that achieves

$$x(0) = \xi, \quad x(T) = \zeta.$$

- the shorter the time $T > 0$ is, the larger the magnitude and the shorter the support of $u(t)$ should be.
- The shape of $u(t)$ may approach to something like the Dirac's delta when T approaches to zero.
- This is actually impossible.

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T-controllable set

- We usually assume the following limitation on $u(t)$:

$$|u(t)| \le 1, \quad \forall t \in [0, T].$$

- A control that satisfies this constraint is called an admissible control.
- The admissible control can be characterized by the T-controllable set.

T-controllable Set

Fix $T > 0$. The set of initial states that can be steered to the origin by some admissible control $u(t)$, $0 \le t \le T$ is called the T-controllable set. We denote this set by $\mathcal{R}(T)$.

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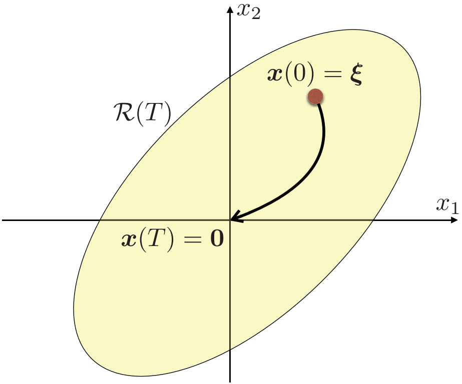
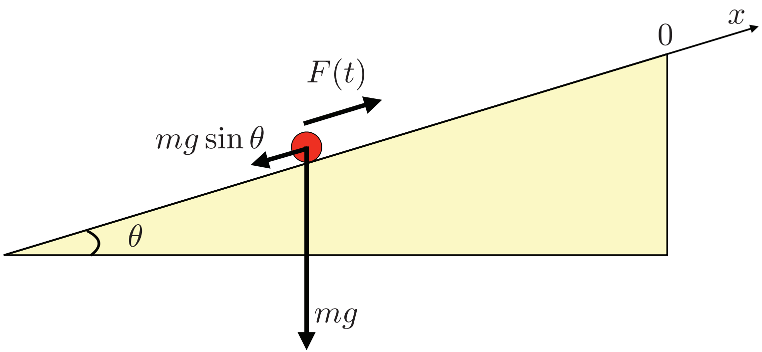
T-controllable set

Theorem

For any $T > 0$, the T-controllable set $\mathcal{R}(T)$ is a bounded, closed, and convex set. Also, if $T_1 < T_2$ then $\mathcal{R}(T_1) \subset \mathcal{R}(T_2)$.



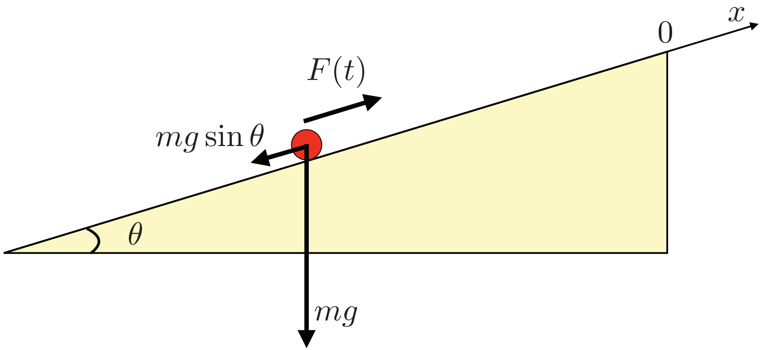
Example



- Move the mass from $x(0) = -\xi$ to $x(T) = 0$ by the force $F(t) \leq 1$.
- ODE

$$m\ddot{x}(t) = F(t) - mg\sin\theta.$$

Example



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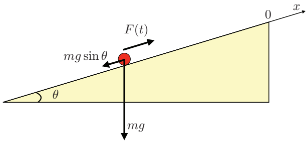

- Let

$$T^* \triangleq \sqrt{\frac{2m\xi}{1 - mg\sin\theta}}.$$

- We observe that
  - If $T < T^*$ there is no admissible control.
  - If $T = T^*$ there is just one admissible control $F(t) = 1$, $t \in [0, T^*]$.
  - If $T > T^*$ there is at least one admissible control:

$$F(t) = \begin{cases} 1, & \text{if } 0 \le t \le T^*, \\ 0, & \text{if } T^* < t \le T. \end{cases}$$

- T^* is called the minimum time.

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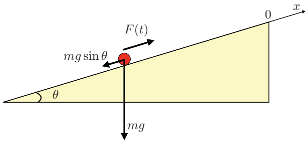

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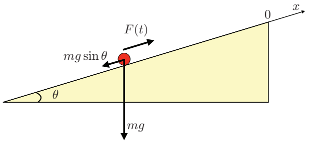

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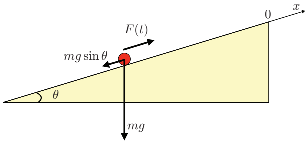

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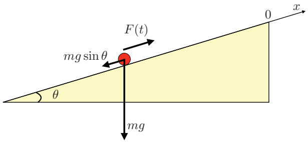

- Let

$$T^* \triangleq \sqrt{\frac{2m\xi}{1 - mg\sin\theta}}.$$

- We observe that
  - If $T < T^*$ there is no admissible control.
  - If $T = T^*$ there is just one admissible control $F(t) = 1$, $t \in [0, T^*]$.
  - If $T > T^*$ there is at least one admissible control

$$F(t) = \begin{cases} 1, & \text{if } 0 \le t \le T^*, \\ 0, & \text{if } T^* < t \le T, \end{cases}$$

- T^* is called the minimum time.

Controllable set

- The minimum time $T^*(\xi)$ from the initial state ξ to the origin is defined as

$$T^*(\xi) \triangleq \inf\{T \geq 0 : \xi \in \mathcal{R}(T)\}.$$

- Is the minimum time finite?
- Define the controllable set

$$\mathcal{R} \triangleq \bigcup_{T>0} \mathcal{R}(T).$$

- If $\xi \in \mathcal{R}$, then $T^*(\xi) < \infty$.
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- Even if the system is controllable, the controllable set $\mathcal{R}$ may not be $\mathbb{R}^d$.
  - If the system is controllable, and the matrix A is stable, that is,
    $$\lambda(A) \subset \mathbb{C}_- \triangleq \{z \in \mathbb{C} : \mathrm{Re}\, z \leq 0\},$$
    then $\mathcal{R} = \mathbb{R}^d$.
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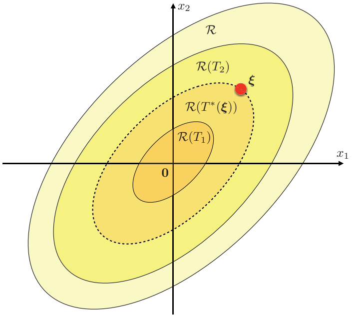

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- Set of all feasible controls is denoted by $\mathcal{U}(T, \xi)$.
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$$\underset{u}{\text{minimize}} \; T \quad \text{subject to} \quad u \in \mathcal{U}(T, \xi). \qquad (\star)$$

- The solution of this optimization is called the minimum-time control or time-optimal control.

Theorem

Assume $T^*(\xi) < \infty$. Then there exists a minimum-time control $u^* \in \mathcal{U}(T^*(\xi), \xi)$. Moreover, for any $T > T^*(\xi)$, $\mathcal{U}(T, \xi)$ is non-empty.

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find an admissible control u (i.e. $\|u\|_\infty \leq 1$) that achieves

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and minimizes the following cost function

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- For the minimum-time control we have $L(u) = 1$ (i.e. $J(u) = T$), and T is not fixed.

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- The Hamiltonian for the optimal control problem is defined by

$$H^{\eta}(\boldsymbol{x}, \boldsymbol{p}, u) \triangleq \boldsymbol{p}^{\top}(A\boldsymbol{x} + \boldsymbol{b}u) + \eta L(u)$$

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- Assume that an optimal control u^* of the optimal control problem (OPT) exists.
- Let us denote by $x^*(t)$ the optimal state with the optimal control $u^*(t)$, that is,

$$x^*(t) \triangleq e^{At}\xi + \int_0^t e^{A(t-\tau)} \boldsymbol{b} u^*(\tau) d\tau, \quad \forall t \in [0, T].$$

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(non-triviality condition) The abnormal multiplier η and the optimal costate $\boldsymbol{p}^*$ satisfy the non-triviality condition:

$$|\eta| + \|\boldsymbol{p}^*\|_\infty > 0.$$

(canonical equation) The following canonical equations hold

$$\begin{aligned} \dot{\boldsymbol{x}}^*(t) &= A\boldsymbol{x}^*(t) + \boldsymbol{b}u(t), \\ \dot{\boldsymbol{p}}^*(t) &= -A^\top \boldsymbol{p}^*(t), \quad \forall t \in [0, T]. \end{aligned}$$

The differential equation for $\boldsymbol{p}(t)$ is called the adjoint equation.

(minimum condition) The optimal control $u^*(t)$ minimizes Hamiltonian at each time $t \in [0, T]$. That is,

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(consistency) Hamiltonian satisfies

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where c is a constant independent of t. If T is not fixed (as in the minimum-time control), then

$$H^{\eta}\big(\boldsymbol{x}^*(t), \boldsymbol{p}^*(t), u^*(t)\big) = 0, \quad \forall t \in [0, T].$$

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$$u^*(t) = \arg\min_{u\in[-1,1]} \boldsymbol{p}^*(t)^\top \boldsymbol{b} u = -\mathrm{sgn}(\boldsymbol{p}^*(t)^\top \boldsymbol{b}),$$

$$\mathrm{sgn}(a) = \begin{cases} 1, & a > 0 \\ -1, & a < 0 \end{cases}$$

$$\mathrm{sgn}(a) \in [-1, 1], \quad a = 0.$$

Minimum-time control

- Minimum-time control

$$u^*(t) = -\mathrm{sgn}(\boldsymbol{p}^*(t)^\top \boldsymbol{b}),$$

- If $\boldsymbol{p}^*(t)^\top \boldsymbol{b} = 0$, then $u^*(t)$ cannot be uniquely determined.

Lemma

If $(A, \boldsymbol{b})$ is controllable, then the function $\boldsymbol{p}^*(t)^\top \boldsymbol{b}$ is not zero for almost all $t \in [0, T^*(\xi)]$.

- If $(A, \boldsymbol{b})$ is controllable, the the minimum-time control is a piecewise constant function that takes values ± 1, which is called a bang-bang control.

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- For the minimum-time control problem, we have the following existence and uniqueness theorems.

Theorem (Existence)

If the initial state ξ is in the controllable set $\mathcal{R}$ then a minimum-time control exists.

Theorem (Uniqueness)

Assume that $(A, \boldsymbol{b})$ is controllable. Then the minimum-time control is (if it exists) unique.

Corollary

Assume that $(A, \boldsymbol{b})$ is controllable and A is stable. Then for any $\xi \in \mathbb{R}^d$, the minimum-time control $u^* \in \mathcal{U}(\xi)$ uniquely exists.

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Table of Contents

Rocket control problem

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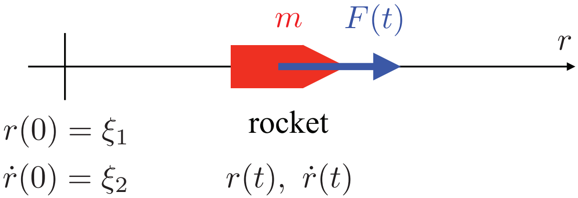
where $\boldsymbol{x}(t) = [r(t), \dot{r}(t)]^\top$, $u(t) = F(t)$, and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix}$$

- Since $(A, \boldsymbol{b})$ is controllable and A is stable (the eigs are $0, 0$), there uniquely exists the minimum-time control $u^*(t)$ for any initial state $\xi \in \mathbb{R}^2$.



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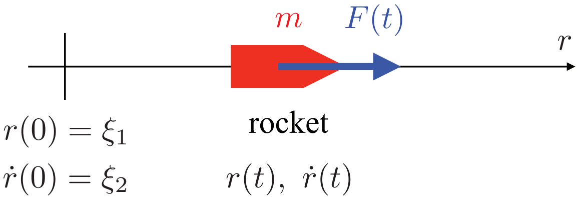

Minimum-time control of rocket

- The Hamiltonian for the minimum-time control is given by

$$H^{\eta}(\boldsymbol{x}, \boldsymbol{p}, u) = \boldsymbol{p}^{\top}(A\boldsymbol{x} + \boldsymbol{b}u) + \eta = p_1 x_2 + p_2 u + \eta.$$

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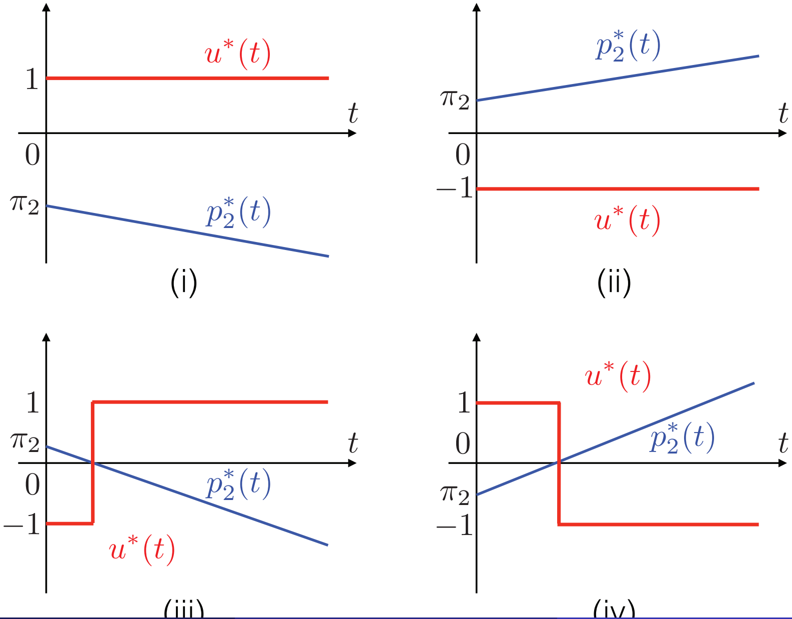

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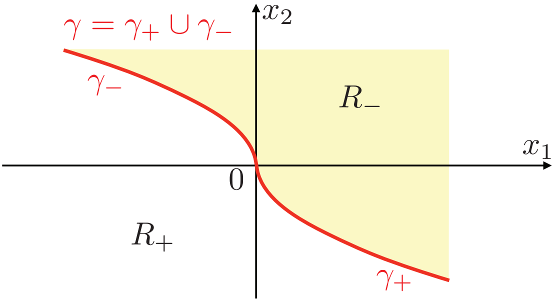
- The minimum-time control is bang-bang:

$$u^*(t) = \begin{cases} 1, & \text{if } \boldsymbol{x}(t) \in \gamma_+ \cup R_+ \setminus \{\mathbf{0}\}, \\ -1, & \text{if } \boldsymbol{x}(t) \in \gamma_- \cup R_- \setminus \{\mathbf{0}\}, \\ 0, & \text{if } \boldsymbol{x}(t) = \mathbf{0}. \end{cases}$$



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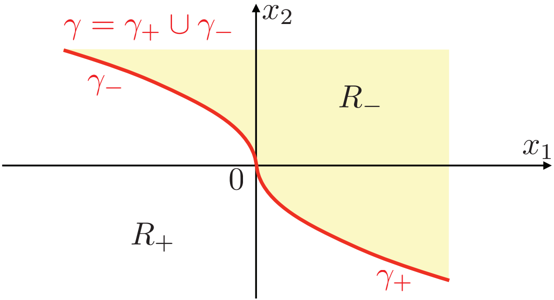

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Conclusion

- A dynamical system is modeled by a differential equation called the state-space equation.
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