Sparsity Methods for Systems and Control

Applications of Sparse Representation

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Curve fitting

- Two-dimensional noisy data:

$$\mathcal{D} = \{(t_1, y_1), (t_2, y_2), \dots, (t_m, y_m)\}.$$

- We find a function, which is not necessarily a polynomial, $y(t)$ that has the best fit to the m-point data.
- We here consider splines in particular.

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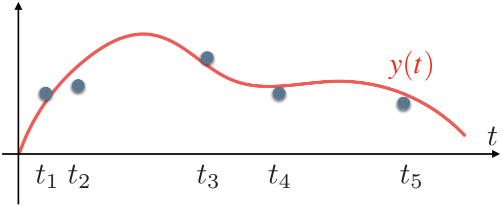

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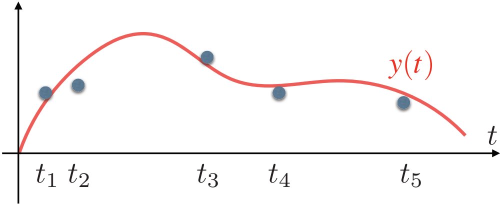

Spline curve fitting problem

Find a smooth function y over $[0, T]$ with $\ddot{y} \in L^2(0, T)$ that minimizes

$$J(y) = \sum_{i=1}^{m} |y(t_i) - y_i|^2 + \lambda \int_0^T |\ddot{y}(t)|^2 dt,$$

- The first term is for the fidelity of curve fitting to the data, and the second term is for the smoothness of the curve.
- This is an infinite-dimensional problem since we seek a function in a function space, not a finite number of parameters.
- This can be reduced to a finite-dimensional optimization problem, by using techniques in Hilbert space theory.

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Optimal control problem

- The problem:

$$\underset{y}{\text{minimize}} \quad \sum_{i=1}^{m} |y(t_i) - y_i|^2 + \lambda \int_0^T |\ddot{y}(t)|^2 dt,$$

- This can be described as an optimal control problem.
- Let

$$x_1(t) \triangleq y(t), \quad x_2(t) \triangleq \dot{y}(t), \quad u(t) \triangleq \ddot{y}(t).$$

- Then the problem can be rewritten as

$$\underset{u \in L^2(0,T)}{\text{minimize}} \quad \sum_{i=1}^{m} |y(t_i) - y_i|^2 + \lambda \int_0^T |u(t)|^2 dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad y = x_1.$$

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General problem

- A general problem

$$\begin{aligned} \underset{u \in L^2(0,T)}{\text{minimize}} \quad & \sum_{i=1}^{m} |y(t_i) - y_i|^2 + \lambda \int_0^T |u(t)|^2 dt \\ \text{subject to} \quad & \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad y(t) = \boldsymbol{c}^\top \boldsymbol{x}(t), \quad t \in [0, T] \\ & \boldsymbol{x}(0) = \boldsymbol{0} \end{aligned}$$

- Define

$$l(\tau, t) \triangleq \begin{cases} \boldsymbol{c}^\top e^{A(t-\tau)} \boldsymbol{b}, & \text{if } 0 \le t \le \tau \\ 0, & \text{otherwise} \end{cases}$$

and

$$\phi_i(t) \triangleq l(t, t_i), \quad i = 1, 2, \dots, m.$$

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Sampled value as inner product

- Then, from the solution of

$$\dot{x}(t) = Ax(t) + bu(t), \quad y(t) = c^\top x(t), \quad x(0) = \mathbf{0}$$

we have

$$y(t_i) = \langle \phi_i, u \rangle_{L^2} = \int_0^T \phi_i(t) u(t) dt, \quad i = 1, 2, \dots, m.$$

- Now the problem becomes

$$\underset{u \in L^2(0,T)}{\text{minimize}} \sum_{i=1}^{m} \left| \langle \phi_i, u \rangle_{L^2} - y_i \right|^2 + \lambda \int_0^T |u(t)|^2 dt.$$

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Optimization problem in Hilbert space

- We further rewrite the problem as

$$\begin{aligned} \underset{u \in L^2(0,T)}{\text{minimize}} \quad & \sum_{i=1}^{m} |z_i - y_i|^2 + \lambda \int_0^T |u(t)|^2 dt \\ \text{subject to} \quad & z_i = \langle \phi_i, u \rangle_{L^2}, \quad i = 1, 2, \ldots, m. \end{aligned}$$

- Define a new Hilbert space $H = L^2(0,T) \times \mathbb{R}^m$ with inner product

$$\left\langle \begin{bmatrix} u \\ z \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_H \triangleq w^\top z + \int_0^T u(t)v(t)dt, \quad u, v \in L^2(0,T), \ z, w \in \mathbb{R}^m$$

- Then the cost function becomes

$$\|r - p\|_H^2 = \langle r - p, r - p \rangle_H,$$

where $r \triangleq (u, z)$, $z = [z_1, \ldots, z_m]^\top$, $p \triangleq (0, y)$, $y = [y_1, \ldots, y_m]^\top$.

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Constraint

- Consider a closed linear subspace M of H defined by

$$M \triangleq \left\{ \begin{bmatrix} u \\ z \end{bmatrix} \in H : z_i = \langle \phi_i, u \rangle_{L^2} \right\},$$

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Optimization as projection

- Finally the optimization problem is now rewritten as

$$\underset{\boldsymbol{r}\in H}{\text{minimize}}\ \|\boldsymbol{r}-\boldsymbol{p}\|_H^2 \ \text{ subject to } \ \boldsymbol{r}\in M.$$

- The minimizer is given by the projection of $\boldsymbol{p}\in H$ onto the closed linear subspace $M\subset H$. That is,

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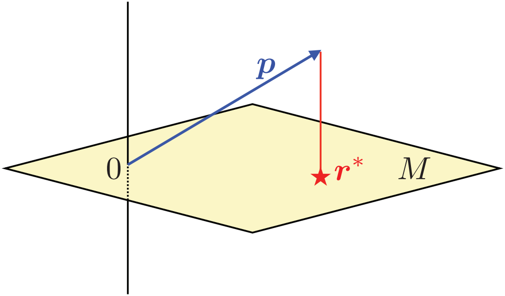

Projection theorem

- Let $M^\perp$ denote the orthogonal complement of M in H. That is,

$$M^\perp \triangleq \left\{ \begin{bmatrix} v \\ \boldsymbol{w} \end{bmatrix} : \left\langle \begin{bmatrix} v \\ \boldsymbol{w} \end{bmatrix}, \begin{bmatrix} u \\ \boldsymbol{z} \end{bmatrix} \right\rangle_H = 0, \ \forall \begin{bmatrix} u \\ \boldsymbol{z} \end{bmatrix} \in M \right\}.$$

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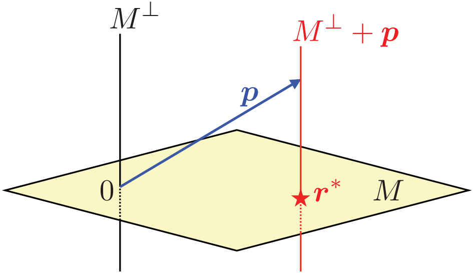

Orthogonal projection to M

- Characterize $(M^\perp + \boldsymbol{p}) \cap M$.
- $M^\perp + \boldsymbol{p}$: Take $(v, \boldsymbol{w}) \in M^\perp$. Then, for any $(u, \boldsymbol{z}) \in M$, we have

$$\begin{aligned} 0 &= \big\langle (v, \boldsymbol{w}), (u, \boldsymbol{z}) \big\rangle_H \\ &= \boldsymbol{w}^\top \boldsymbol{z} + \lambda \int_0^T v(t)u(t)dt \\ &= \sum_{i=1}^m w_i \langle \phi_i, u \rangle_{L^2} + \lambda \langle v, u \rangle_{L^2} \\ &= \left\langle \sum_{i=1}^m w_i \phi_i + \lambda v, u \right\rangle_{L^2} \end{aligned}$$

  - Note that $M = \big\{ (u, \boldsymbol{z}) \in H : z_i = \langle \phi_i, u \rangle_{L^2} \big\}$

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holds for any $u \in L^2(0, T)$, and hence

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$$v = -\frac{1}{\lambda} \sum_{i=1}^{m} w_i \phi_i .$$

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$$M^\perp = \left\{ \begin{bmatrix} -\frac{1}{\lambda}\sum_{i=1}^{m} w_i\phi_i \\ \boldsymbol{w} \end{bmatrix} : \boldsymbol{w} \in \mathbb{R}^m \right\}$$

- Adding $\boldsymbol{p} = (0, \boldsymbol{y})$ to this, we have
$$M^\perp + \boldsymbol{p} = \left\{ \begin{bmatrix} -\frac{1}{\lambda}\sum_{i=1}^{m} w_i\phi_i \\ \boldsymbol{w} + \boldsymbol{y} \end{bmatrix} : \boldsymbol{w} \in \mathbb{R}^m \right\}.$$

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- Optimal solution $\boldsymbol{r}^* = (u^*, \boldsymbol{z}^*) \in (M^\perp + \boldsymbol{p}) \cap M$.
- Since $\boldsymbol{r}^* \in M$, we have

$$z_i^* = \langle \phi_i, u^* \rangle_{L^2}, \quad i = 1, 2, \dots, m. \qquad (\star)$$

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- Insert $(\star\star)$ into $(\star)$.

Orthogonal projection to M

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$$w_i + y_i = \left\langle \phi_i, -\frac{1}{\lambda}\sum_{j=1}^m w_j \phi_j \right\rangle_{L^2} = -\frac{1}{\lambda}\sum_{j=1}^m w_j \langle \phi_i, \phi_j \rangle_{L^2}.$$

- Define the Gram matrix

$$G \triangleq \begin{bmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_1, \phi_2 \rangle & \dots & \langle \phi_1, \phi_m \rangle \\ \langle \phi_2, \phi_1 \rangle & \langle \phi_2, \phi_2 \rangle & \dots & \langle \phi_2, \phi_m \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_m, \phi_1 \rangle & \langle \phi_m, \phi_2 \rangle & \dots & \langle \phi_m, \phi_m \rangle \end{bmatrix}.$$

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Optimal solution

- Finally, we have

$$\boldsymbol{w} = -\lambda(\lambda I + G)^{-1}\boldsymbol{y}.$$

- Note that w is the coefficient vector of the optimal solution u^*:

$$u^* = -\frac{1}{\lambda}\sum_{i=1}^{m} w_i \phi_i$$

- Therefore, the optimal solution u^* is given by

$$u^* = \sum_{i=1}^{m} \alpha_i^* \phi_i$$

where $\boldsymbol{\alpha}^* = (\lambda I + G)^{-1}\boldsymbol{y}$.

- The optimal curve is then given by

$$y^*(t) = \int_0^t \int_0^\tau u^*(s)\,ds\,d\tau.$$

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- From $u = \sum_i z_i \phi_i$, we have

$$\langle \phi_i, u \rangle_{L^2} = \left\langle \phi_i, \sum_{j=1}^{m} z_j \phi_j \right\rangle_{L^2} = \sum_{j=1}^{m} z_j \langle \phi_i, \phi_j \rangle_{L^2}$$

- And hence

$$\sum_{i=1}^{m} \left| \langle \phi_i, u \rangle_{L^2} - y_i \right|^2 = \sum_{i=1}^{m} \left| \sum_{j=1}^{m} z_j \langle \phi_i, \phi_j \rangle_{L^2} - y_i \right|^2 = \|G\boldsymbol{z} - \boldsymbol{y}\|^2.$$

- Also, we have

$$\lambda \int_0^T |u(t)|^2 dt = \lambda \sum_{i=1}^{m} \sum_{j=1}^{m} z_i z_j \langle \phi_i, \phi_j \rangle_{L^2} = \lambda \boldsymbol{z}^\top G \boldsymbol{z}.$$

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- Therefore, the cost function becomes

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- Now we formulate the sparse representation of z:

$$\underset{z \in \mathbb{R}^m}{\text{minimize}} \|Gz - y\|^2 + \lambda z^\top G z + \rho \|z\|_1.$$

- This can be solved by FISTA.

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Table of Contents

Control problem

- Discrete-time system:

$$\boldsymbol{x}[k+1] = A\boldsymbol{x}[k] + \boldsymbol{b}u[k], \quad k = 0, 1, 2, \ldots, n-1,$$

Control problem

Assume that the initial state $x[0] = \xi$ is given by state observation. Find a control sequence $\{u[0], u[1], \ldots, u[n-1]\}$ such that the control drives the state $x[k]$ from $x[0] = \xi$ to the origin, that is,

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Feasible set

- State equation

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x[n] &= A^n \xi + \sum_{i=0}^{n-1} A^{n-1-i} b u[i] = A^n \xi + \Phi u \\
\Phi &\triangleq \begin{bmatrix} A^{n-1}b & A^{n-2}b & \ldots & Ab & b \end{bmatrix} \\
u &\triangleq \big[u[0], u[1], \ldots, u[n-1]\big]^\top.
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- The set of feasible solutions (feasible set):

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\mathcal{U}(n, \xi) \triangleq \left\{ u \in \mathbb{R}^n : x[n] = A^n \xi + \Phi u = 0 \right\}.
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Feasible set

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$$\begin{aligned} \boldsymbol{x}[k+1] &= A\boldsymbol{x}[k] + \boldsymbol{b}u[k], \quad k = 0, 1, 2, \ldots, n-1 \\ \boldsymbol{x}[0] &= \boldsymbol{\xi}. \end{aligned}$$

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Feasible set

- If $n \geq d$ and the plant is controllable, then the feasible set $\mathcal{U}(n, \xi)$ is non-empty for any initial state $\xi \in \mathbb{R}^d$.
- We then consider optimal control:

$$\underset{u \in \mathbb{R}^n}{\text{minimize}} \ J(u) \text{ subject to } u \in \mathcal{U}(n, \xi),$$

where

$$J(u) = \sum_{k=0}^{n-1} L(x[k], u[k]).$$

  - LQ control: $L(x, u) = x^\top Q x + r|u|^2$
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ℓ^1 optimal control

- The ℓ^1 optimal control problem:

$$\underset{\boldsymbol{u}\in\mathbb{R}^n}{\text{minimize}} \;\; \|\boldsymbol{u}\|_1 \text{ subject to } A^n\boldsymbol{\xi} + \Phi\boldsymbol{u} = \boldsymbol{0},$$

- This can be solved by Douglas-Rachford splitting algorithm or ADMM!
- We can easily extend the finite-horizon control to feedback control using model predictive control (MPC).
  - Lyapunov stability is guaranteed if A is non-singular. [Nagahara, Østergaard, Quevedo, 2016]
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Summary

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- By Hilbert space theory (the projection theorem), the optimization is reduced to a finite-dimensional problem (a quadratic optimization problem).
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