Sparsity Methods for Systems and Control

Curve Fitting and Sparse Optimization

Masaaki Nagahara[1]

[1]The University of Kitakyushu
nagahara@ieee.org

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Minimum-ℓ^2 solution

- Linear equation: $y = \Phi x$
  - $y \in \mathbb{R}^m$ is given
  - $\Phi \in \mathbb{R}^{m \times n}$ is given
  - $x \in \mathbb{R}^n$ is unknown
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Solution of ℓ^2 optimization problem

- Lagrangian

$$L(x,\lambda) = \frac{1}{2}x^\top x + \lambda^\top(\Phi x - y).$$

- Differentiate L by x

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}x^\top x + \lambda^\top \Phi x\right) = x + \Phi^\top \lambda.$$

- The stationary point equation

$$x^* + \Phi^\top \lambda^* = \mathbf{0}. \quad \text{(i)}$$

- Also, x^* satisfies the equation $\Phi x = y$, we have

$$\Phi x^* = y \quad \text{(ii)}$$

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- Inserting (i) into (ii) gives

$$-\Phi\Phi^\top \boldsymbol{\lambda}^* = \boldsymbol{y}$$

- Since $\mathrm{rank}(\Phi) = m$, $\Phi\Phi^\top$ is invertible and

$$\boldsymbol{\lambda}^* = -(\Phi\Phi^\top)^{-1}\boldsymbol{y}.$$

- Finally, we obtain the solution from (i) as

$$\boldsymbol{x}^* = \Phi^\top(\Phi\Phi^\top)^{-1}\boldsymbol{y}.$$

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Polynomial curve fitting

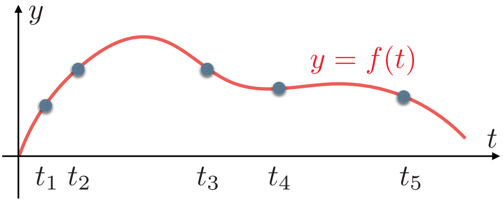
- Two-dimensional data:

$$\mathcal{D} = \{(t_1, y_1), (t_2, y_2), \ldots, (t_m, y_m)\}.$$

- Polynomial curve fitting

$$y = f(t) = a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0.$$

to find coefficients $a_0, a_1, \ldots, a_{n-1}$ with which the polynomial curve has the best fit to the m-point data.



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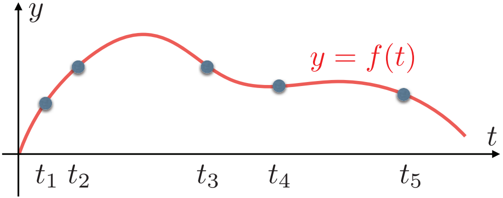

Interpolating polynomial

- The polynomial curve

$$y = f(t) = a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0.$$

goes through the data points.

- linear equations with for unknown coefficients

$$\begin{aligned} a_{n-1}t_1^{n-1} + a_{n-2}t_1^{n-2} + \cdots + a_1 t_1 + a_0 &= y_1, \\ a_{n-1}t_2^{n-1} + a_{n-2}t_2^{n-2} + \cdots + a_1 t_2 + a_0 &= y_2, \\ &\cdots \\ a_{n-1}t_m^{n-1} + a_{n-2}t_m^{n-2} + \cdots + a_1 t_m + a_0 &= y_m. \end{aligned}$$

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\end{aligned}$$

Vandermonde's matrix

- Define

$$\Phi \triangleq \begin{bmatrix} t_1^{n-1} & t_1^{n-2} & \dots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \dots & t_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ t_m^{n-1} & t_m^{n-2} & \dots & t_m & 1 \end{bmatrix} \in \mathbb{R}^{m\times n},$$

$$\boldsymbol{x} \triangleq \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} \in \mathbb{R}^n, \quad \boldsymbol{y} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m.$$

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- Φ is called Vandermonde's matrix.

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- If $m = n$, then the determinant of Φ is given by

$$\det(\Phi) = \prod_{1 \leq i < j \leq m} (t_i - t_j) = (t_1 - t_2)(t_1 - t_3) \cdots (t_{m-1} - t_m).$$

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MATLAB Simulation

- Data:

t	1	2	3	...	14	15
y	2	4	6	...	28	30

$\rightarrow y = 2t$

- Result:

```
x =

   2.274746684520826e-24
  -5.565271161256770e-21
   9.137367918505765e-19
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  -3.658098129966092e-16
  -1.608088662230500e-15
   3.569367024169878e-14
  -6.021849685566849e-13
   5.346834086594754e-13
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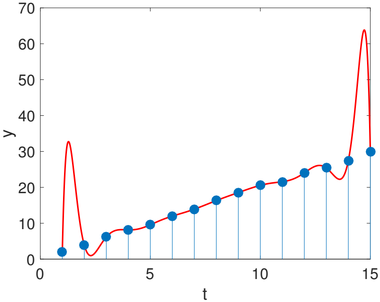
- Add Gaussian noise with mean 0 and variance 0.5^2 to the data y.
- Curve fitting via Vandermonde's matrix inversion $\Phi^{-1} y$.



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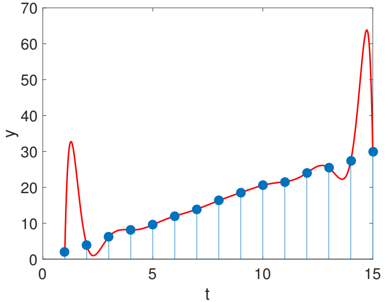

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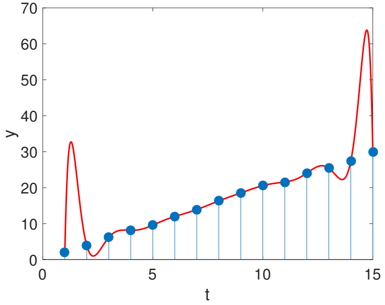

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Least squares method

- The order of the polynomial was too large.
- We can first assume a first-order polynomial $y = a_1 t + a_0$.
- The line does not interpolate the noisy data.
- Find a polynomial that minimizes the ℓ^2 error:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \frac{1}{2}\|\Phi x - y\|_2^2,$$

- This is called the least squares method.

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The least squares solution

- The error function

$$E(x) = \frac{1}{2}\|\Phi x - y\|_2^2 = \frac{1}{2}(\Phi x - y)^\top(\Phi x - y)$$
$$= \frac{1}{2}x^\top\Phi^\top\Phi x - y^\top\Phi x + \frac{1}{2}y^\top y.$$

- Φ is $m \times n$ with $m > n$ (a tall matrix).
- If $t_i \neq t_j$ for all i, j such that $i \neq j$, then Φ has full column rank, and $\Phi^\top\Phi > 0$ (positive definite).
- The minimizer x^* satisfies

$$\frac{\partial E}{\partial x}(x^*) = (\Phi^\top\Phi)x^* - \Phi^\top y = \mathbf{0}.$$

- The solution

$$x^* = (\Phi^\top\Phi)^{-1}\Phi^\top y.$$

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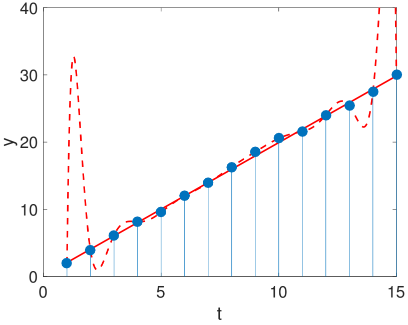
MATLAB Simulation

- Assume the curve is a first-order polynomial.
- The data is noisy.



Another example

- How can we know a proper order of the polynomial?
- It is often difficult.
- Data set

$$\mathcal{D} = \{(t_1, y_1), (t_2, y_2), \dots, (t_m, y_m)\},$$

where $y_i = \sin(t_i) + \epsilon_i$, with $t_i = i - 1$, $i = 1, 2, \dots, 11$ and ϵ_i is Gaussian noise.

- The data:

t_i	0	1	2	3	4	5
y_i	-0.0343	1.0081	0.8326	0.4047	-0.7585	-0.9285
t_i	6	7	8	9	10	
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- How can we know a proper order of the polynomial?
- It is often difficult.
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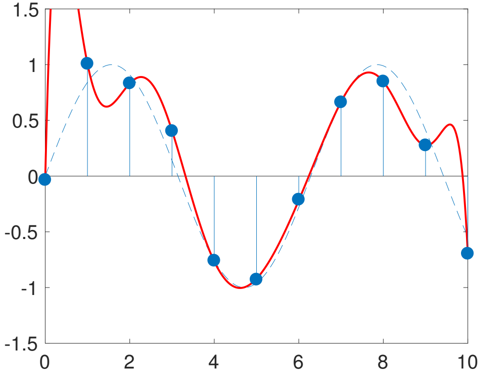
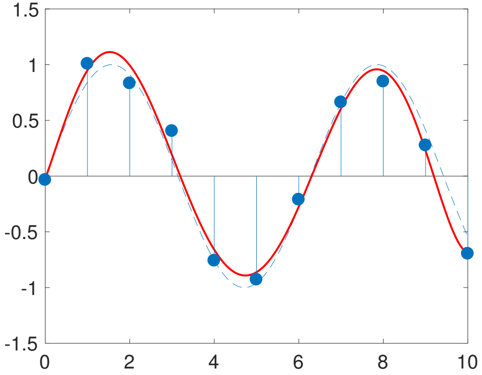
- 10th order interpolating polynomial



Another example

- 6th order polynomial by the least squares method



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- What is the difference?
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$$x_{10} = \begin{bmatrix} -0.0343 \\ 16.2400 \\ -38.0984 \\ 37.8369 \\ -20.2842 \\ 6.5035 \\ -1.3100 \\ 0.1677 \\ -0.0133 \\ 0.0006 \\ -0.0000 \end{bmatrix}, \quad x_6 = \begin{bmatrix} -0.0260 \\ 1.0636 \\ 0.3067 \\ -0.5225 \\ 0.1426 \\ -0.0146 \\ 0.0005 \end{bmatrix},$$

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- Idea: keep the polynomial order high and reduce the norm of the coefficient vector
- That is,

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; \frac{1}{2}\|\Phi x - y\|_2^2 + \frac{\lambda}{2}\|x\|_2^2.$$

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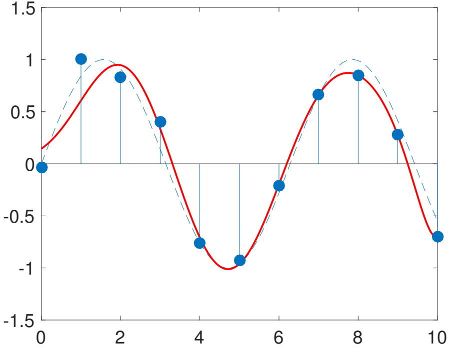

Polynomial curve fitting: summary

- data

$$\mathcal{D} = \{(t_1, y_1), (t_2, y_2), \ldots, (t_m, y_m)\}.$$

- polynomial

$$y = f(t) = a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0,$$

Problem	Matrix size	Opt. problem	Solution
min ℓ^2 norm	$m < n$	$\min_{x} \frac{1}{2}\|x\|_2^2$ s.t. $y = \Phi x$	$\Phi^\top(\Phi\Phi^\top)^{-1}y$
least squares (LS)	$m > n$	$\min_{x} \frac{1}{2}\|y - \Phi x\|_2^2$	$(\Phi^\top\Phi)^{-1}\Phi^\top y$
regularized LS	any m and n	$\min_{x} \frac{1}{2}\|y - \Phi x\|_2^2 + \frac{\lambda}{2}\|x\|_2^2$	$\Phi^\top(\lambda I + \Phi\Phi^\top)^{-1}y$ $= (\lambda I + \Phi^\top\Phi)^{-1}\Phi^\top y$

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An example

- 80th-order polynomial

$$y = -t^{80} + t.$$

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- There are infinitely many interpolating polynomials
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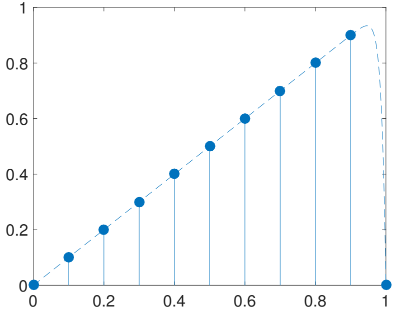

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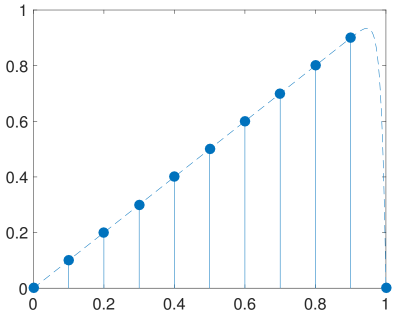

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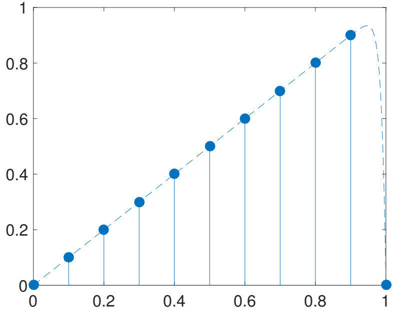

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- Consider the ℓ^0 optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; \|x\|_0 \quad \text{subject to} \quad \Phi x = y.$$

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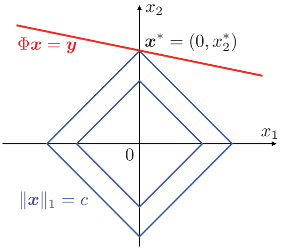

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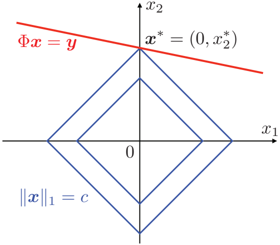

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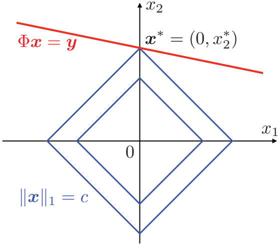

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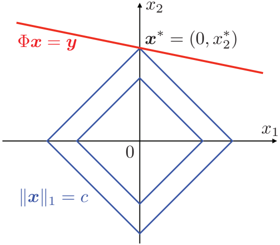

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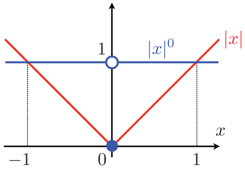

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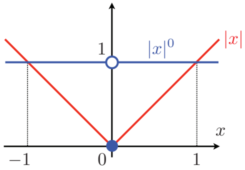

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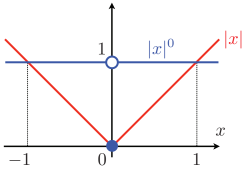

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ℓ^1 Regularization (LASSO)

- For noisy data, we consider ℓ^0 regularization:

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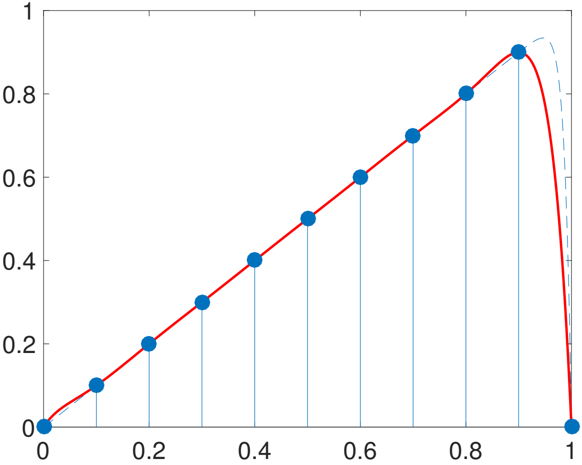
- 10th-order interpolating polynomial



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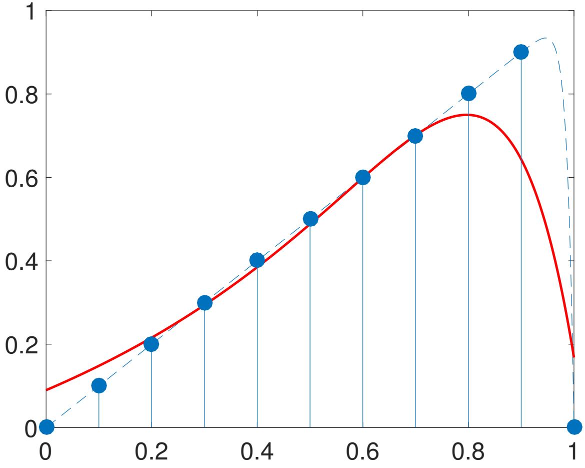

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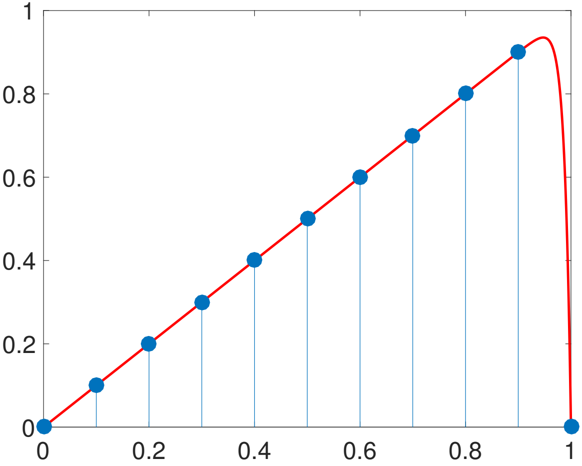
- ℓ^1 optimization (80th order)



Summary

- Curve fitting is formulated as an optimization problem to choose one solution among (infinitely many) candidates.
- Regularization is used for avoiding over fitting.
- Sparse optimization is reduced to ℓ^1 optimization, which is convex and efficiently solved by numerical optimization.

Summary

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