Optimal Design of $\Delta\Sigma$ Modulators via Generalized KYP Lemma

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Abstract: In this article, we propose a new design method of $\Delta\Sigma$ modulators. We propose an optimal design to shape the frequency response of the noise transfer function (NTF). Generalized KYP (Kalman-Yakubovic-Popov) lemma is used to reduce our optimization to a linear matrix inequality. Design examples are illustrated to show effectiveness of our method.

Keywords: Delta-sigma modulator, generalized KYP lemma, quantization.

1. INTRODUCTION

$\Delta\Sigma$ *modulators* [1] are widely used in AD (Analog-to-Digital) and DA (Digital-to-Analog) converters, in which high performance can be obtained with coarse quantizers.

A fundamental issue in designing $\Delta\Sigma$ modulators is *noise shaping* in the frequency domain [1]. A usual solution to this is to insert accumulator(s) in the feedback loop to attenuate the gain of the noise transfer function (NTF) in low frequencies. This methodology looks like PID (Proportional-Integral-Derivative) control [2], in which the performance of the designed system depends on the amount of experiences of the designer. That is, the conventional design is of an ad hoc nature.

Let us consider a general $\Delta\Sigma$ modulator shown in Fig. 1. In this modulator, Q is a quantizer and $H = [H_1, H_2]$ is a linear filter with 2 inputs and 1 output. The filter H_1 shapes the signal transfer function (STF) from the input u to the output y to have a unity gain in the frequency band of interest. On the other hand, the filter H_2 eliminates the in-band quantization noise by shaping the NTF.

To shape *optimally* the NTF in the frequency band of interest, say $[0, \Omega]$, the NTF zero optimization [1] can be used. This method is to minimize the normalized noise power, given by the integral of the squared magnitude of the NTF over $[0, \Omega]$. On the other hand, we minimize the maximum of the gain of the NTF in $[0, \Omega]$. This is related to a minimax optimization (or an H^∞ one), and more effective than the NTF zero optimization in terms of uniform attenuation of the frequency response over the band. We have proposed an H^∞ optimization in [3], in which we have to choose a suitable weighting function to obtain a good performance. On the other hand, we propose in this article more useful method with no weighting function, by generalized Kalman-Yakubovic-Popov (KYP) lemma [4]. Then the optimization can be reduced to one with a linear matrix inequality (LMI). The

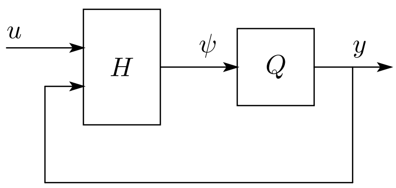

Fig. 1 $\Delta\Sigma$ modulator

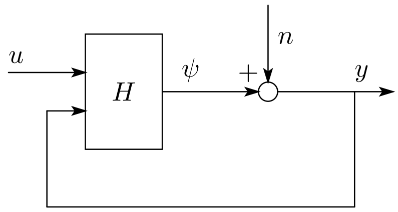

Fig. 2 Linearized model for $\Delta\Sigma$ modulator

idea to apply generalized KYP lemma to $\Delta\Sigma$ modulator design is also proposed in [5], in which they assume one-bit quantizer for Q and optimize the average power of the reconstruction error in low frequencies. In contrast to this approach, our optimization is for quantization noise shaping, which is more familiar to engineers and researchers in this area. Moreover, the zeros of the NTF can be assigned arbitrarily on the unit circle in the complex plane with linear matrix equality (LME), and the stability condition can be described by an H^∞ constraint norm of the NTF, which can be an LMI. That is, the proposed method can be described by LMI's and LME's, which can be solved effectively by numerical computation softwares such as MATLAB. Design examples show effectiveness of our method.

2. CHARACTERIZATION OF LOOP FILTERS

In this section, we first characterize all $H(z)$'s which stabilize the linearized model shown in Fig. 2. A necessary condition that a $\Delta\Sigma$ modulator is stable is that its linearized model is internally stable. Note that the converse is generally not true, that is, even if the linearized model is stable, the nonlinear system in Fig. 1 can be unstable. A stability condition for the nonlinear system is discussed in 3.3

We first characterize the filter $H(z)$ which internally stabilizes the linearized feedback system. All stabilizing filters are characterized as follows [3].

Lemma 1: The linearized feedback system in Fig. 2 is well-posed and internally stable if and only if

$$\begin{aligned} &H_1(z) = \frac{R_1(z)}{1 + R_2(z)}, \quad H_2(z) = \frac{R_2(z)}{1 + R_2(z)}, \\ &R_1(z) \in \mathcal{S}, \quad R_2(z) \in \mathcal{S}', \end{aligned} \tag{1}$$

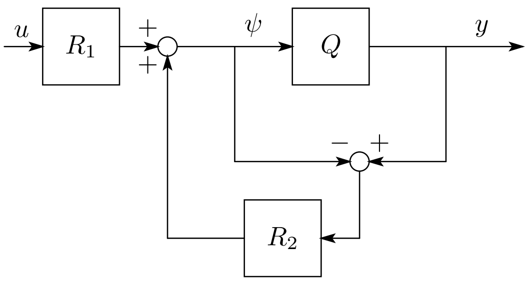

Fig. 3 Error-feedback structure of $\Delta\Sigma$ modulator with design parameters $R_1 \in \mathcal{S}$ and $R_2 \in \mathcal{S}'$.

where $\mathcal{S}$ is the set of all stable, causal, real-rational transfer functions, and $\mathcal{S}' := \{R \in \mathcal{S} : R \text{ is strictly causal}\}$. By using these parameters $R_1 \in \mathcal{S}$ and $R_2 \in \mathcal{S}'$, the STF and NTF are given by

$$T_{\text{STF}}(z) = R_1(z), \quad T_{\text{NTF}}(z) = 1 + R_2(z),$$

and the input/output equation of the system in Fig. 2 is given by

$$y = R_1 u + (1 + R_2)n. \tag{2}$$

By (2), the structure of the $\Delta\Sigma$ modulator with the design parameters $R_1 \in \mathcal{S}$ and $R_2 \in \mathcal{S}'$ is shown in Fig. 3. This structure, called error-feedback structure, is often applied in the digital loops required in $\Delta\Sigma$ DA converters [1]. By this block diagram, we can interpret the filter R_1 as a pre-filter to shape the frequency response of the input signal, and R_2 as a feedback gain for the quantization noise $Q\psi - \psi$.

3. OPTIMAL LOOP FILTER DESIGN VIA LINEAR MATRIX INEQUALITIES AND EQUALITIES

In this section, we propose an optimal design of the loop filter $H(z)$ by using the parameterization in Lemma 1.

3.1 Optimal noise shaping via generalized KYP lemma

For simplicity, we assume $R_1(z) = 1$. This means that the STF is assumed to be allpass. Then our problem is formulated as follows.

Problem 1: Given Ω $(0 < \Omega < \pi)$ and $\gamma > 0$, find $R_2(z) \in \mathcal{S}'$ which satisfies

$$\sup_{\omega \in [0,\Omega]} |T_{\text{NTF}}(e^{j\omega})| < \gamma. \tag{3}$$

In implementation, finite impulse response (FIR) filters are often preferred, and hence we assume that $R_2(z)$ is FIR, that is, we set

$$R_2(z) = \sum_{k=0}^{N} \alpha_k z^{-k}, \quad \alpha_0 = 0.$$

Note that $R_2(z)$ is always in $\mathcal{S}'$. We then introduce state-space matrices $\{A, B, C(\alpha)\}$, such that $R_2(z) = C(\alpha)(zI - A)^{-1}B$, where $\alpha = \begin{bmatrix}\alpha_0 & \alpha_1 & \dots & \alpha_N\end{bmatrix}$,

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $C(\alpha) = [\alpha_N, \alpha_{N-1}, \dots, \alpha_1]$. Then the inequality (3) can be described as a linear matrix inequality (LMI) by using the generalized KYP lemma [4].

Theorem 1: The inequality (3) holds if and only if there exist symmetric matrices $Q > 0$ and P such that

$$\begin{bmatrix} M_1(P,Q) & M_2(P,Q) & C(\alpha)^\top \\ M_2(P,Q) & M_3(P,\gamma^2) & 1 \\ C(\alpha) & 1 & -1 \end{bmatrix} < 0,$$

where

$$M_1(P,Q) = A^\top PA + QA + A^\top Q - P - 2Q\cos\Omega,$$
$$M_2(P,Q) = A^\top PB + QB,$$
$$M_3(P,\gamma^2) = B^\top PB - \gamma^2.$$

By Theorem 1, the optimal coefficients $\alpha_1, \dots \alpha_N$ can be obtained efficiently by standard optimization softwares, e.g., MATLAB (See [6]).

3.2 NTF zeros

To ensure perfect reconstruction of the DC input level, and to reduce low-frequency tones, $T_{\text{NTF}}(z)$ should have zeros at $z = 1$, or the frequency $\omega = 0$ [1]. A similar requirement is in bandpass $\Delta\Sigma$ modulator; we set NTF zeros at a given frequency ω_0, or $z = e^{\pm j\omega_0}$. The zeros of $T_{\text{NTF}}(z)$ can be assigned by linear equations (linear constraints) of $\alpha_1, \dots, \alpha_N$. Define $n(z) := z^N + \sum_{k=1}^{N} \alpha_k z^{N-k}$. Then, $T_{\text{NTF}}(z)$ has M zeros at $z = z_0$ if and only if

$$\left.\frac{d^k n(z)}{dz^k}\right|_{z=z_0} = 0, \quad k = 0, 1, \dots, M-1,$$

where $\frac{d^0 n(z)}{dz^0} := n(z)$. The LMI with these linear constraints can be also effectively solved.

3.3 Stability condition

The linearized model Fig. 2 is useful for analyzing the noise shaping properties of $\Delta\Sigma$ modulators. The stability of $\Delta\Sigma$ modulators, however, should be analyzed by considering their nonlinear behaviors. To analyze the stability, the H^∞ norm of $T_{\text{NTF}}(z)$ is available. For the stability of binary $\Delta\Sigma$ modulators, the following criterion (Lee criterion) is widely used [7], [1]:

$$\|T_{\text{NTF}}\|_\infty < 1.5. \tag{4}$$

Note that this is not a sufficient nor necessary condition for the stability. For multi-bit modulators with M-step quantizer, the following is a sufficient condition for the stability [8], [1]:

$$\|h\|_1 \le M + 2 - \|u\|_\infty, \tag{5}$$

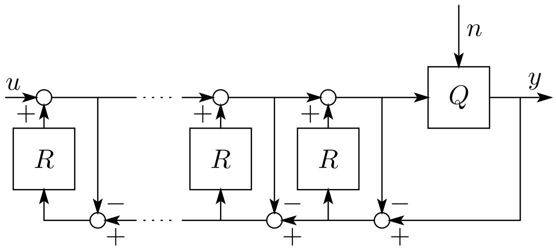

Fig. 4 Cascade of Error Feedback

where h is the impulse response of $T_{\rm NTF}(z)$ and u is the input signal. Let N denote the order of $T_{\rm NTF}(z)$. Then, we have the following relation [9]:

$$\|h\|_1 \leq (2N+1)\|T_{\rm NTF}\|_\infty.$$

By combining this with (5), we have another stability condition.

$$\|T_{\rm NTF}\|_\infty \leq \frac{1}{2N+1}(M+2-\|u\|_\infty). \qquad (6)$$

From the conditions (4) and (6), attenuation of $\|T_{\rm NTF}\|_\infty$ helps the stability. Therefore, we add the following stability constraints to the design of modulators:

$$\|T_{\rm NTF}\|_\infty < C,$$

where $C > 0$ is a constant (e.g., by Lee criterion, $C = 1.5$). This inequality is also reducible to LMI [6] and easily combined with the LMI optimization mentioned above.

4. CASCADE OF ERROR FEEDBACK FOR HIGH-ORDER MODULATORS

Assume that $R_1 = 1$ and $R_2 = R$. To design a high-order modulator, we can use a cascade of the error feedback modulator in Fig. 3. The proposed cascade structure is shown in Fig. 4. By using this structure, we have

$$T_{\rm STF}(z) = 1, \quad T_{\rm NTF}(z) = (1+R)^M,$$

where M denotes the number of filters R. If $R \in \mathcal{S}'$, then the linearized feedback system is stable. An advantage of this structure is that the number of taps of R can be reduced, and hence the implementation is much easier than a filter with a large number of taps. This structure can be applied to $\Delta\Sigma$ DA converters.

To satisfy the stability condition $\|T_{\rm NTF}\|_\infty < C$, the filter R is designed to limit

$$\|1+R\|_\infty < C^{1/M}.$$

If this is satisfied, we have

$$\|T_{\rm NTF}\|_\infty \leq \|1+R\|_\infty^M < C,$$

by the sub-multiplicative property of the H^∞ norm. In this section, we show examples of designing $\Delta\Sigma$ modulators by the proposed method.

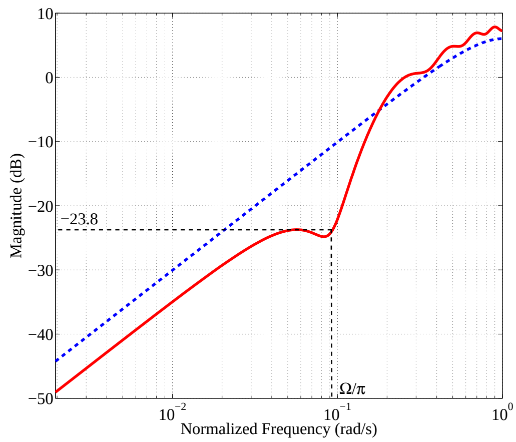

Fig. 5 Frequency response of $T_{\rm NTF}$: proposed (solid line) and conventional (dash)

5. DESIGN EXAMPLES

5.1 $\Delta\Sigma$ Modulator with DC zero

We here design the filter $R_2(z)$ which is an FIR filter with 12 taps, and set $R_1(z) = 1$. The cut-off frequency Ω is $3\pi/32$. The NTF $1 + R_2(z)$ is designed to have a zero at $z = 1$ to attenuate DC noise most, and also to satisfy the stability condition $\|T_{\rm NTF}\|_\infty < 1.5$ (these constraints can be described as linear matrix equality and inequality, see [3]). By this optimization, we obtain the minimum value of $\gamma = 6.48 \times 10^{-2}$ (-23.8 [dB]). Fig. 5 shows $T_{\rm NTF}$'s by the proposed method and the first order $\Delta\Sigma$ modulator. The $T_{\rm NTF}$ of our design shows a lower gain in the low frequency and a higher gain in the high frequency. The frequency response in Fig. 5 is that of the linearized system shown in Fig. 2. To see the nonlinear effect in the quantizer, we simulate responses against sinusoidal waves with various frequencies. The reconstruction filter after the $\Delta\Sigma$ modulator is chosen to be H^∞ optimal one proposed in [3]. Fig. 6 shows NSR (Noise-to-Signal Ratio) against sinusoidal waves. The NSR shows that our $\Delta\Sigma$ modulator shows a better response than the conventional one in all frequencies. Fig. 7 and Fig. 8 shows outputs respectively of proposed and conventional $\Delta\Sigma$ converters against a sinusoidal wave.

5.2 Higher order modulator

We here show a design example of a higer order modulator with cascade structure in Fig. 4. We set $R_1 = 1$, and $R_2(z) = R(z)$ be an FIR filter with 32 taps. The cutoff frequency Ω is set to be $\pi/32$. The FIR filter $R(z)$ is designed by using the LMI in Theorem 1, with the stability condition $\|T_{\rm NTF}\|_\infty < 1.5$. The number of cascades M is 4, that is, the order of the modulator is $32 \times 4 = 128$. We also design a modulator by the NTF zero optimization [1] which minimize the normalized noise power of the NTF. This modulator is designed by the MATLAB function `synthesizeNTF` in the Delta-Sigma Toolbox [1], with the order of $T_{\rm NTF}$ is 4, the over sampling ratio (OSR) is 32, and $\|T_{\rm NTF}\|_\infty < 1.5$.

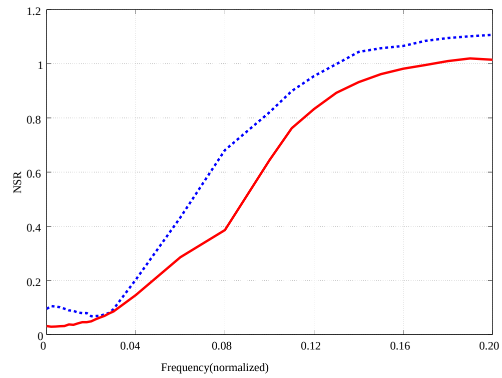

Fig. 6 NSR against sinusoidal waves: proposed (solid) and conventional (dash)

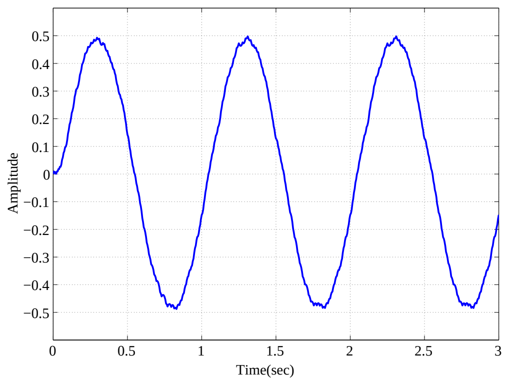

Fig. 7 Output against sinusoidal wave (proposed)

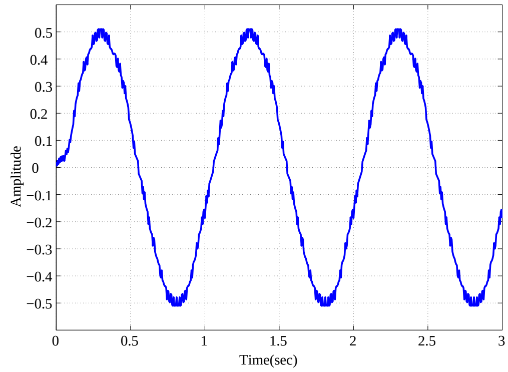

Fig. 8 Output against sinusoidal wave (conventional)

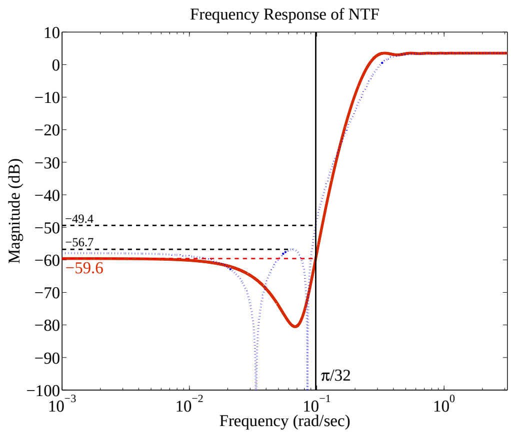

Fig. 9 NTF's: proposed (solid) and the NTF zero optimization (dots)

Fig. 9 show the frequency responses of the proposed modulator and the NTF zero optimized one. By this figure, the gain of the proposed NTF is uniformly attenuated over $[0, \pi/32]$ while the conventional one shows a peak in the band. The maximal difference between two gains is about 10 [dB].

6. CONCLUSION

In this paper, we have propose a new design method of $\Delta\Sigma$ modulators. We have characterized the all stabilizing loop filters for linearized model. Based on this, we have formulated our problem of noise shaping in the frequency domain. By using generalized KYP lemma, our design is reducible to an LMI optimization. Assignment of NTF zeros and stability condition are described by LMI's and LME's, respectively. Design examples have shown effectiveness of our method.

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