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Optimal Wavelet Expansion via Sampled-Data H^∞ Control Theory

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Abstract: Wavelet expansion of an L^2 signal requires the L^2 inner product of the original signal and a scaling function. In digital signal processing, it is common to use sampled data of continuous-time signals instead of the inner product. This however causes a large reconstruction error, called "wavelet crime." We therefore design a causal system which produces an approximation of the inner product via sampled-data H^∞ control theory. We then make extensions to a multi-rate and a multi-wavelet case. By numerical examples, we show the effectiveness of the proposed method.

Keywords: wavelet, sampled-data control, prefiltering, signal reconstruction.

1. INTRODUCTION

Wavelet [3], [9] is an effective tool for analyzing continuous-time signals. In wavelet theory, a continuous-time signal is expressed as a linear combination of waves (scaling and wavelet functions) which are localized in time and frequency. This expression, called wavelet expansion, gives us time-frequency analysis of signals, which is efficient in analyzing in particular non-stationary signals. For example, the image format JPEG2000 is based on wavelet expansion and has higher compression rate and better quality than the conventional JPEG based on Fourier expansion [10].

Wavelet expansion of a continuous-time signal begins with calculating coefficients of each wave contained in the signal. When the signal is in L^2 (the Lebesgue space of square integrable functions on $\mathbb{R}$), each coefficient, called a wavelet coefficient, is equal to the L^2 inner product of the signal and a scaling function. The computation of the inner product requires all values of the signal on $\mathbb{R}$. In real situation, however, it is usual that only the sampled data of the original signal are available. In this case, it is impossible to obtain the exact values of the coefficients. Therefore, approximation is essential for computing the coefficients in real signal processing. The roughest approximation is to use the sampled data themselves as the coefficients. This is often used in real, by which arises a large reconstruction error, called "wavelet crime [9]."

To prevent this crime, prefiltering methods have been proposed [4], [5] in the situation where only the sampled data are available. They are based on the sampling theorem by Shannon, which assumes that the original continuous-time signals are fully band-limited up to the Nyquist frequency. However, this assumption fails to consider practical signals such as rectangle or triangle waves.

In recent papers [8], [6], new approaches to this problem has been developed in the framework of sampled-data control theory. By using this theory, we can obtain the optimal system taking account of the frequency characteristic of the original analog signals.

Motivated by these studies, we consider in the article the approximation problem via sampled-data control theory. In [6], only the analog characteristic of the input signals is considered. On the other hand, we also consider the characteristic of the reconstructed analog signals. The design problem is formulated as a sampled-data H^∞ optimization with a generalized hold. We also make extensions to a multi-rate and a multi-wavelet case. By numerical examples, we show the effectiveness of the proposed method.

2. WAVELET EXPANSION

In this section, we discuss wavelet expansion. First, we introduce multi-resolution analysis [7] which plays an essential role in wavelet theory. Multi-resolution analysis is defined as an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces in L^2, and each subspace V_j has a Riesz basis $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ where

$$\phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k).$$

In this definition, the function $\phi \in L^2$ is called as a scaling function. Next, consider the orthogonal complement $W_j := V_{j+1} \ominus V_j$. It can be shown that there exists a function $\psi \in L^2$ such that each subspace W_j has a Riesz basis $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$, where

$$\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k).$$

Wavelet expansion of a continuous-time signal begins with specifying a resolution J. Let f be in L^2 and P_J be the orthogonal projection in L^2 onto the closed subspace $V_J \subset L^2$. By using the basis $\{\phi_{J,k}\}_{k\in\mathbb{Z}}$ in V_J, the projection of f onto V_J can be expressed as

$$P_J f = \sum_{k\in\mathbb{Z}} c_J(k)\phi_{J,k}. \quad (1)$$

On the other hand, corresponding to the direct sum representation

$$V_J = V_{j_0} \oplus W_{j_0} \oplus \cdots \oplus W_{J-1},$$

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the projection can be also decomposed as

$$P_J f = \sum_{k\in\mathbb{Z}} c_{j_0}(k)\phi_{j_0,k} + \sum_{j=j_0}^{J-1}\sum_{k\in\mathbb{Z}} d_j(k)\psi_{j,k}. \quad (2)$$

This is the wavelet expansion of f at the resolution J, and the coefficients $\{c_{j_0}, d_{j_0}, \ldots, d_{J-1}\}$ are called wavelet coefficients. Since equations (1) and (2) are different expressions of the same signal $P_J f$, the wavelet coefficients can be calculated efficiently from only $\{c_J(k)\}_{k\in\mathbb{Z}}$ by a filter bank. From the above discussion, it can be seen that the accuracy of the wavelet expansion at the resolution J depends on that of the projection onto V_J. In other words, it is necessary to accurately compute the coefficients $\{c_J(k)\}_{k\in\mathbb{Z}}$.

In particular, if $\{\phi_{J,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_J, the coefficients $\{c_J(k)\}_{k\in\mathbb{Z}}$ are given as follows:

$$\begin{aligned} c_J(k) &= (f, \phi_{J,k})_{L^2} \\ &= 2^{J/2}\int_{-\infty}^{\infty} f(t)\phi(2^J t - k)dt. \end{aligned}$$

However, as explained in Section 1, this inner product cannot be computed exactly in practice. In order to overcome this difficulty, we propose in the next section a design method of systems, which produce approximated coefficients using only sampled data.

3. COMPUTATION OF COEFFICIENTS BY H^∞ OPTIMIZATION

In this section, we propose a design method of systems which produce an approximation of wavelet coefficients of a continuous-time signal from only its sampled data. The problem is formulated as a sampled-data H^∞ optimization one. In this section, it is assumed that the resolution J is equal to 0 and that the signal we aim to expand is in $L^2[0,\infty)$. Moreover, we make an important assumption about the scaling function.

Assumption 1: *There exists $m > 0$ such that the support of the scaling function ϕ is in $[0, m]$.*

First, we characterize the wavelet coefficient c_0 of the signal $f \in L^2[0,\infty)$. From the projection theorem, the coefficient sequence $c_0 \in \ell^2$ is characterized as the solution $c \in \ell^2$ of the following optimization problem:

$$\|f - \mathcal{H}_\phi c\|_{L^2[0,\infty)} \to \min, \quad (3)$$

where $\mathcal{H}_\phi$ is defined as

$$\mathcal{H}_\phi : \ell^2 \to L^2[0,\infty) : c \mapsto \sum_{k\geq 0} c(k)\phi_{0,k}. \quad (4)$$

Next, let us consider the error system in Fig. 1. On the lower path, the exogenous signal $w \in L^2[0,\infty)$ is band limited by a transfer function F which has the frequency characteristic of the continuous-time signals we

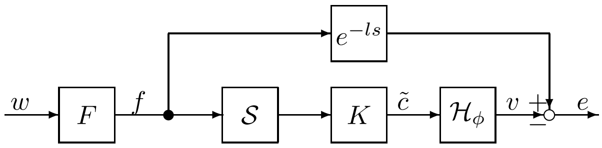


Fig. 1 Error system

aim to expand, and the signal f is the model signal to be expanded. This signal f is then sampled by the ideal sampler

$$\mathcal{S} : FL^2[0,\infty) \to \ell^2,$$
$$(\mathcal{S}f)(k) := f(k), \quad k = 0, 1, 2, \ldots.$$

where

$$FL^2[0,\infty) := \{Fw : w \in L^2[0,\infty)\}.$$

The sampled data are processed by the digital filter K, and then transformed into a continuous-time signal $v = \mathcal{H}_\phi\tilde{c} \in V_0$.

Since the wavelet coefficient c_0 of the signal f is characterized by (3), the signal $\tilde{c}$ in Fig. 1 approximates c_0 if the error $e = f - \mathcal{H}_\phi\tilde{c}$ is small. Hence, in order to obtain a digital filter which produces the approximation of the coefficient c_0, it should be designed to make the error e to be as small as possible.

Note that the signal f is delayed by $e^{-ls} : f \mapsto f(\cdot - l)$ on the upper path. Without this delay, the reconstruction process becomes anti-causal one, and hence it is desirable to take l to be no less than m.

Now, we formulate the design problem as follows.

Problem 1: *Given a stable and strictly proper transfer function F, a delay $l > 0$, a scaling function ϕ, and a prespecified bound $\gamma > 0$, find a digital filter K satisfying $\|T_{ew}\|_\infty < \gamma$, where $\|T_{ew}\|_\infty$ is the $L^2[0,\infty)$ induced norm of the operator T_{ew} from w to e in Fig. 1.*

Invoking fast-sample/fast-hold approximation [11], we can reduce this sampled-data H^∞ optimization problem to a discrete-time one. This discrete-time problem is a finite dimensional one by the assumption 1, and can be effectively solved by MATLAB routines. Hence, by bisection search, we can easily obtain a sub-optimal digital filter minimizing $\|T_{ew}\|_\infty$.

4. EXTENSIONS

4.1 Higher resolution

In the above section, we have shown an approximation method to compute wavelet coefficients at the resolution 0. Then, what should we do in order to compute higher resolution coefficients?

Let us consider the error system in Fig. 2. Compared with Fig. 1, the difference is that the sampled data of the original signal f are first upsampled by $\uparrow 2^J$. The upsampler $\uparrow 2^J$ is defined as follows:

$$\uparrow 2^J : \{x(k)\}_{k=0}^{\infty} \to \{x(0), \underbrace{0, 0, \ldots, 0}_{2^J - 1}, x(1), 0, \ldots\}.$$

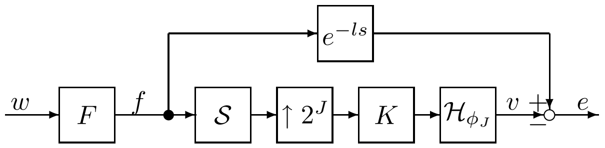


Fig. 2 Error system: higher resolution case

The upsampling operation is implemented by inserting $2^J - 1$ equidistant zero-valued samples between two consecutive samples of $x(k)$ before the sampling rate is multiplied by the factor 2^J. Then the signal is processed by the digital filter K, which estimates the data between the original samples. Then the estimated (virtually) higher resolution signal is transformed to a continuous signal by the following generalized hold:

$$\mathcal{H}_{\phi_J} : \ell^2 \to L^2[0, \infty) : c \mapsto \sum_{k \geq 0} c(k)\phi_{J,k}, \tag{5}$$

which acts with it's sampling period $1/2^J$.

The optimal digital filter K is obtained by minimizing $L^2[0, \infty)$ induced norm of the operator from w to e in Fig. 2. This minimization is also a multi-rate sampled-data H^∞ optimization. The multi-rate system is equivalently transformed into a single-rate one by discrete-time lifting [1] or polyphase representation [9]. Then, by fast-sample/fast-hold approximation, the problem is reduced to a discrete-time one.

4.2 Multi-wavelet

Multi-wavelet is an extension of the conventional wavelet and uses a wavelet basis generated by several scaling functions called multi-scaling functions. By this extension, there arises great flexibility in the design of a wavelet, and desirable properties such as symmetry or orthonormality of scaling functions can be realized simultaneously [2].

In multi-wavelet, the multi-resolution analysis is constructed by multi-scaling functions $\phi^{(1)}, \ldots, \phi^{(r)} \in L^2$, and the subspace V_J has a Riesz basis $\{\phi^{(1)}_{J,k_1}, \ldots, \phi^{(r)}_{J,k_r} : k_1, \ldots, k_r \in \mathbb{Z}\}$, where

$$\phi^{(l)}_{J,k} := 2^{J/2}\phi^{(l)}(2^J \cdot -k).$$

Then, the orthogonal projection of a signal $f \in L^2[0, \infty)$ onto the closed subspace $V_J \subset L^2$ has the form

$$P_J f = \sum_{l=1}^{r} \sum_{k \in \mathbb{Z}} c^{(l)}_J(k)\phi^{(l)}_{J,k}(k).$$

As in the scalar case, these coefficients $c^{(1)}_J, \ldots, c^{(r)}_J$ are called as wavelet coefficients of the signal f at the resolution J, and can be characterized as the solution $c^{(1)}, \ldots, c^{(r)} \in \ell^2$ of the following optimization problem:

$$\left\| f - \sum_{l=1}^{r} \mathcal{H}_{\phi^{(l)}} c^{(l)} \right\|_{L^2[0,\infty)} \to \min, \tag{6}$$

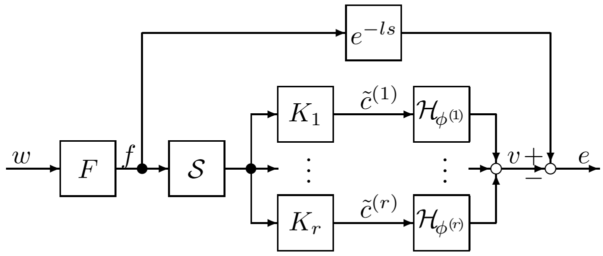


Fig. 3 Error system: multi-wavelet case

where $\mathcal{H}_{\phi^{(l)}}$ is defined in the same way as (4) and (5).

Now, let us consider the error system in Fig. 3. As in Section 3, the resolution J is assumed to be equal to 0. Since the wavelet coefficients are characterized as the solution of the optimization problem (6) similar to the scalar case one (3), we can formulate the design problem in the same way as problem 1.

Since H^∞ optimization is executed in the state space in which MIMO (multi-input multi-output) systems can be treated in the same way as SISO (single-input single-output) systems, the formulated problem can be treated as a sampled-data H^∞ optimization one (in this case, the filter to be designed is a single-input and r-output system), and so can be reduced to a discrete-time finite dimensional one. And hence sub-optimal filters $K_1, \ldots, K_r$ which compute the approximation of the wavelet coefficients $c^{(1)}, \ldots, c^{(r)}$ respectively are obtained.

5. NUMERICAL EXAMPLES

In this section, we present numerical examples. Throughout this section, the frequency characteristic F is taken as follows:

$$F(s) = \frac{1}{(7s + 1)(0.7s + 1)}.$$

5.1 Scalar wavelet

As an example of a scalar case, we consider the orthogonal Daubechies-2 wavelet [3]. We make comparisons between the wavelet crime, the proposed method with $J = 0$, and one with $J = 2$. For the proposed method, the optimal filters are designed with $l = 5$ in both cases.

First, we make a comparison by observing the frequency responses of the error systems shown in Fig. 1 ($J = 0$) or Fig. 2 ($J = 2$). Fig. 4 shows the frequency responses. In this figure, the error frequency response of the wavelet crime reconstruction (i.e., $K = 1$) is optimized by varying the delay l so that the H^∞ norm is minimized. It can be seen that the errors by the proposed method are both smaller than the wavelet crime in a wide frequency range. In low frequency range, the wavelet crime reconstruction shows a better response. It follows that the error in wavelet crime appear particularly in a middle and high frequency range.

We can also observe that, by taking the resolution J higher than 0, we can obtain more accurate reconstruction. This shows effectiveness of the use of the interpo-

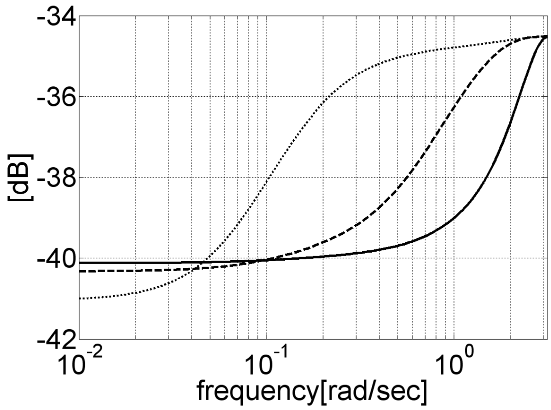


Fig. 4 Frequency responses of the error systems: wavelet crime (dotted), proposed with $J = 0$ (dashed), and $J = 2$ (solid)

lator. Note that the sampling period in the sampler is not changed in the case of $J = 0$ and $J = 2$.

Next, we execute the wavelet expansions of a chirp signal,

$$f(t) = \sin(\omega(t) \cdot t), \quad \omega(t) = \frac{t}{9} + \frac{3}{10}. \tag{7}$$

This signal is a typical non-stationary sinusoid signal whose frequency increases linearly with respect to time t. The reconstructed signals are shown in Fig. 5. The wavelet crime reconstruction shows a peaked response. In Fig. 6, we illustrate the squared reconstruction errors. In this figure, our reconstruction shows smaller error than the wavelet crime. It should be noticed that the error by the wavelet crime reconstruction increases as time passes. This justifies the error frequency response Fig. 4.

We can conclude from these simulations that

- wavelet crime shows in particular middle- and high-frequency errors,
- sampled-data H^∞ optimization is effective in sampled-data wavelet expansion,
- upsampling is also effective.

5.2 Multi-wavelet

In this subsection, we show an example of a multi-wavelet case using the symmetric and orthogonal multi-wavelet with approximation order 2 proposed in [2]. This multi-wavelet has the multi-scaling functions shown in Fig. 7. With this multi-wavelet, we compare the proposed method with one proposed in [5].

The design parameters are as follows. The reconstruction delay $l = 4$, the number of scaling functions is $r = 2$, and the subspace is V_0 (i.e., $J = 0$).

Fig. 8 shows the frequency responses of the error system shown in Fig. 3. Note that in the conventional case, the reconstruction delay l is equal to $3/2$. In this figure, the proposed method shows the better error frequency response than that of the conventional method, in particular in a middle- and high-frequency range. The maximum difference of the response is about 7 [dB] at around 0.5 [rad/sec].

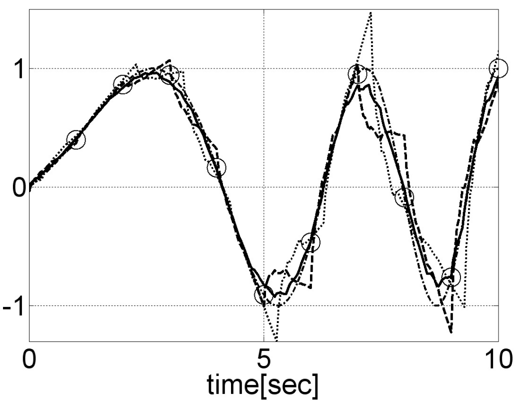


Fig. 5 Time responses of expansions: Source signal (dash-dotted), sampled data (circle), wavelet crime (dotted), proposed with $J = 0$ (dashed), and $J = 2$ (solid)

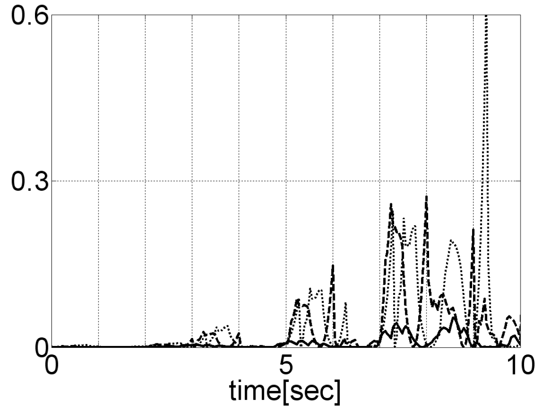
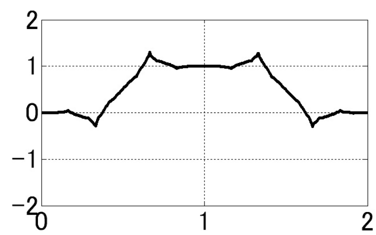
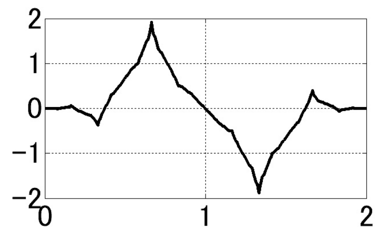


Fig. 6 Time responses of squared expansion errors: wavelet crime (dotted), proposed with $J = 0$ (dashed), and $J = 2$ (solid)



(a) $\phi^{(1)}$



(b) $\phi^{(2)}$

Fig. 7 Multi-scaling functions of Chui-Lian wavelet

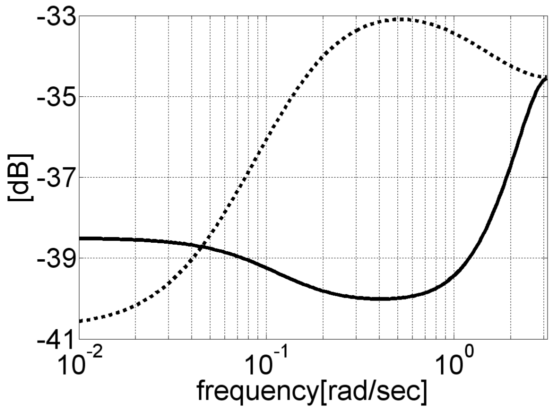


Fig. 8 Frequency responses of the error systems: conventional (dashed) and proposed (solid)

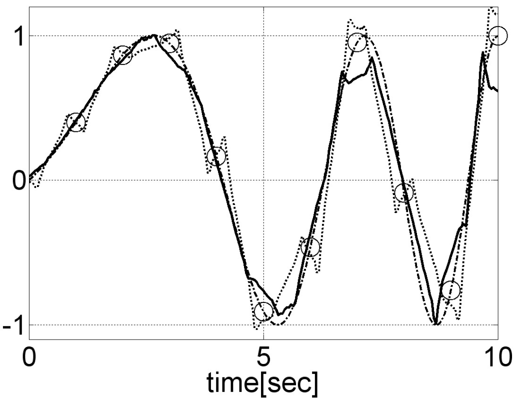


Fig. 9 Time responses of expansions: source signal (dotted), sampled data (circle), conventional (dashed), and proposed (solid)

Then, we simulate the reconstruction with the chirp signal in (7). Fig. 9 shows the time response against the chirp signal. The conventional reconstruction shows a good response at the sampling instants, while in the inter-samples it shows large gaps. On the other hand, the response of the proposed method shows the better response than the conventional one. Fig. 10 shows the squared errors. This figure shows that the error of the proposed method is smaller than the conventional one. From these simulation results, the sampled-data H^∞ optimization is effective also in the multi-wavelet case.

6. CONCLUSION

In this article, a new design method of digital filters which compute approximated wavelet coefficients has been proposed, in the case that only the sampled-data of the original signal are available. The design problem is formulated as an sampled-data H^∞ optimization one. We have shown the problem can be reduced to a discrete-time one by the fast-sample/fast-hold method. The obtained discrete-time H^∞ optimization can be effectively solved by MATLAB routines. This method can be extended to a multi-rate and a multi-wavelet case by using

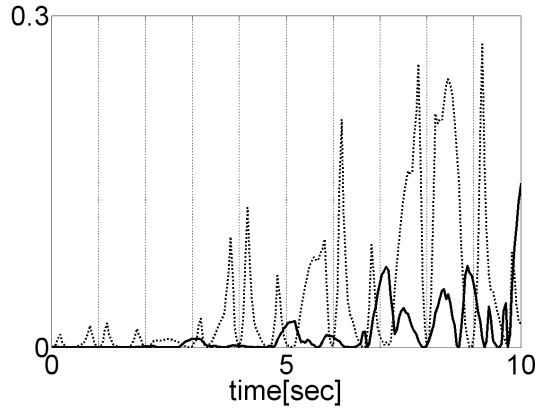


Fig. 10 Time responses of squared expansion errors: conventional (dashed) and proposed (solid)

the discrete-time lifting and an MIMO filter design, respectively. Finally, by numerical examples, the effectiveness of the proposed method have been shown.

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