SICE-ICASE International Joint Conference 2006
Oct. 18-21, 2006 in Bexco, Busan, Korea

Causal Spline Interpolation by H^∞ Optimization

M. Nagahara[†1], T. Wada[†2], and Y. Yamamoto[†3]

[†]Graduate School of Informatics, Kyoto University, Kyoto, Japan
[1]nagahara@ieee.org, [2]twada@acs.i.kyoto-u.ac.jp, [3]yy@i.kyoto-u.ac.jp

Abstract: Spline interpolation systems generally contain non-causal filters, and hence it is difficult to use such systems for real-time processing. Our objective is to design a causal system which approximates spline interpolation. This is formulated as a problem of designing a stable inverse of a system with unstable zeros. For this purpose, we adopt H^∞ optimization. The H^∞-optimal inverse system can be effectively solved by standard MATLAB routines, and hence causal spline interpolation is obtained. A numerical example is presented to illustrate the result.

Keywords: splines, interpolation, H^∞ optimization

1. INTRODUCTION

Signal interpolation has many applications; it is used for curve fitting, signal reconstruction, and sampling rate conversion including resolution conversion of digital images. Many methods for signal interpolation such as polynomial splines [4], [5] and exponential splines [6] have been proposed. Polynomial splines are, in particular, widely used in image processing.

In polynomial (or exponential) spline interpolation, it is assumed that the original signal is a piece-wise polynomial (or exponential) function. Then, intersample values are computed via the Fourier coefficients relative to the spline bases. However, the coefficients are computed by using the future samples, and hence the interpolation system becomes non-causal. The same nature applies to the signal reconstruction by Shannon sampling theorem. In the case of image processing, non-causality is not a restriction, and hence spline interpolation is widely used in image processing. However, it is difficult to use the splines for real-time processing, for example, instrumentation or audio/speech processing.

We therefore propose to design a causal system which approximates polynomial (or exponential) spline interpolation. This is formulated as a problem of designing a stable inverse of a system with unstable zeros. For this purpose, we adopt H^∞ optimization. By this, we obtain the H^∞-optimal inverse system, and hence causal spline interpolation is obtained.

2. SPLINE INTERPOLATION

In this section, we discuss polynomial spline interpolation. Consider $x(t) \in V^N$, $t \in \mathbb{R}$, where V^N is the space of polynomial splines of order N, which is defined as [4],

$$V^N = \left\{ x(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t-k), t \in \mathbb{R}, c \in \ell^2 \right\}.$$

In this equation, $\phi(t)$ is the symmetrical spline of order N, that is,

$$\phi(t) = (\underbrace{\beta^0 * \cdots * \beta^0}_{N+1})(t), \quad \beta^0(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

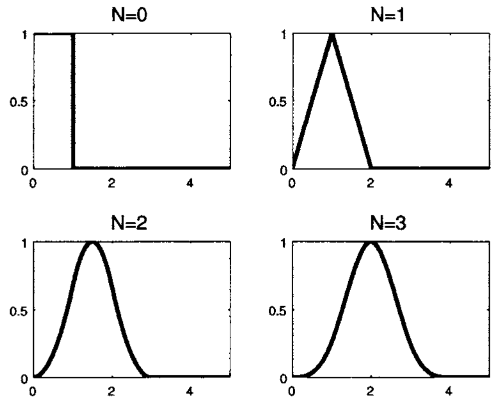

Fig. 1 Polynomial spline of order $N = 0, 1, 2, 3$

Signals in V^N are piecewise polynomials of order N, and when $N \ge 2$, the space V^N is the subset of functions in L^2 that are of class C^{N-1} (continuous functions with continuous derivatives up to order $N-1$) [4]. Fig. 1 shows the polynomial spline $\phi(t)$ of order $N = 0, 1, 2, 3$.

The sampled signal $x(n)$, $n = 0, 1, 2, \ldots$ of $x(t) \in V^N$ with sampling period $T = 1$ is given by

$$x(n) = \sum_{k=-\infty}^{\infty} c(k)\phi(n-k) = (c * \phi)(n). \qquad (1)$$

On the other hand, the fast sampled signal $x_L(n) := x(n/L)$, $n = 0, 1, 2, \ldots$ is given by

$$\begin{aligned} x_L(n) &= x\left(\frac{n}{L}\right) \\ &= \sum_{k=-\infty}^{\infty} c(k)\phi\left(\frac{n}{L} - k\right) \\ &= \sum_{k=-\infty}^{\infty} c(k)\phi_L(n - kL) \qquad (2) \\ &= \sum_{k=-\infty}^{\infty} \{(\uparrow L)c\}(k)\phi_L(n-k) \\ &= (c_L * \phi_L)(n), \end{aligned}$$

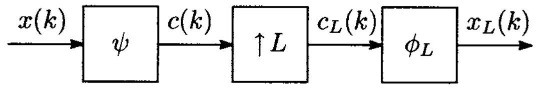

Fig. 2 Spline interpolation

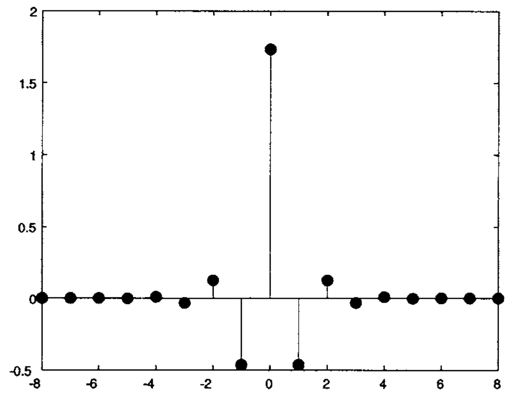

Fig. 3 Impulse response of $\psi(z) = 6/(z + 4 + z^{-1})$

where $\phi_L(n) := \phi(n/L)$ and $c_L := \{(\uparrow L)c\}$. By (1) and (2), the spline interpolation system is composed of two filters ψ and ϕ_L, and the upsampler $\uparrow L$ as shown in Fig. 2, where ψ is a system such that

$$\psi * \phi = I. \tag{3}$$

3. CAUSAL SPLINE INTERPOLATION BY H^∞ OPTIMIZATION

3.1 Non-causal interpolation by decomposition

The Nth-order spline $\phi(t)$ is supported in $[0, N+1)$, and hence the sampled signal $\phi(n)$ or $\phi_L(n)$ is represented as an FIR (finite impulse response) filter. For example, in the case of $N = 3$ (cubic spline), we have

$$\phi(z) = \frac{1}{6}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{6}z^{-3}.$$

By (3), the filter $\psi(z)$ is the inverse $\psi = \phi^{-1}$ and given by

$$\psi(z) = \frac{6z}{1 + 4z^{-1} + z^{-2}}.$$

One of the poles of $\psi(z)$ lies out of the unit circle, and hence the filter $\psi(z)$ is unstable. The same thing is said of the other Nth-order splines [5]. A practical way to implement the filter is to decompose $\psi(z)$ into a cascade of stable causal and anti-causal filters [5]. In the case of the cubic spline, we first shift the impulse response as

$$\phi(z) = \frac{1}{6}z + \frac{2}{3} + \frac{1}{6}z^{-1},$$

and then decompose $\psi(z) = \phi(z)^{-1}$ as

$$\psi(z) = -\frac{6\alpha}{1-\alpha^2}\left(\frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z} - 1\right),$$

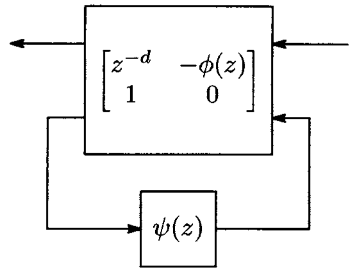

Fig. 4 H^∞ optimization

where $\alpha = \sqrt{3} - 2$. Fig. 3 shows the impulse response of this non-causal filter. Since $|\alpha| < 1$, this is a stable and non-causal IIR (infinite impulse response) filter.

3.2 Causal interpolation by H^∞ optimization

In image processing, causality is not necessary, and the non-causal filter mentioned above is used widely in that field. However, it is difficult to use such non-causal filters for real-time processing, for example, in instrumentation or audio/speech processing. We therefore propose designing a causal filter $\psi(z)$ which approximates the equation (3). Our problem is formulated as follows.

Problem 1: Given a stable transfer function $\phi(z)$ and delay $d \geq 0$, find the causal and stable filter $\psi(z)$ which minimizes

$$J(\psi) = \|z^{-d} - \psi(z)\phi(z)\|_\infty. \tag{4}$$

This is a standard H^∞ optimization problem, and it can be effectively solved by standard MATLAB routines (e.g., robust control toolbox [1]) by using the design block diagram shown in Fig. 4. The optimal filter is generally an IIR one. Since the filter $\psi(z)$ to be designed is linearly dependent upon the error system $z^{-d} - \psi(z)\phi(z)$, the optimal FIR filter with fixed order is obtained by optimization with a linear matrix inequality (LMI).

3.3 FIR filter design via LMI

We here design the filter $\psi(z)$ as an FIR one

$$\psi(z) = \sum_{k=0}^{N} a_k z^{-k}.$$

A state space representation of this FIR filter is given by

$$\psi(z) = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ \hline a_N & \dots & \dots & \dots & a_1 & a_0 \end{array}\right]$$

$$=: \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi(\alpha) & D_\psi(\alpha) \end{array}\right],$$

$$\alpha := \left[\begin{array}{cccc} a_N & \dots & a_1 & a_0 \end{array}\right].$$

Note that the parameter α to be designed is linearly dependent on the matrices $C_\psi(\alpha)$ and $D_\psi(\alpha)$. Set state space representation of $\phi(z)$ and z^{-d} respectively by

$$\phi(z) =: \left[\begin{array}{c|c} A_\phi & B_\phi \\ \hline C_\phi & D_\phi \end{array}\right], \quad z^{-d} =: \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & 0 \end{array}\right].$$

Then, a state space representation of the error system

$$E(z) := z^{-d} - \psi(z)\phi(z)$$

is given by

$$E(z) = \left[\begin{array}{ccc|c} A_\psi & B_\psi C_\phi & 0 & -B_\psi D_\phi \\ 0 & A_\phi & 0 & -B_\phi \\ 0 & 0 & A_d & B_d \\ \hline C_\psi(\alpha) & D_\psi(\alpha)C_\phi & C_d & -D_\psi(\alpha)D_\phi \end{array}\right]$$
$$=: \left[\begin{array}{c|c} A & B \\ \hline C(\alpha) & D(\alpha) \end{array}\right].$$

By this, the parameter α to be designed is linearly dependent on the matrices $C(\alpha)$ and $D(\alpha)$. By using the bounded real lemma, we can describe our design problem as an LMI [7].

Proposition 1: Let γ be a positive number. Then the inequality $\|E(z)\|_\infty < \gamma$ holds if and only if there exist a positive definite matrix $P > 0$ such that

$$\begin{bmatrix} A^TPA - P & A^TPB & C(\alpha)^T \\ B^TPA & -\gamma I + B^TPB & D(\alpha)^T \\ C(\alpha) & D(\alpha) & -\gamma I \end{bmatrix} < 0.$$

4. PERFORMANCE ANALYSIS

In the previous section, we have proposed the H^∞ optimization design of the filter $\phi(z)$ which approximates the equation (3). In this section, we analyze the overall performance of the interpolation system shown in Fig. 2. We first show that the approximation of the equation (3) is proper for increasing the SNR (signal-to-noise ratio) of the interpolation system.

Proposition 2: Assume that $\phi(z)$ and $\psi(z)$ are causal and stable, and $x \in \ell^2$. Let $\widetilde{x}_L$ be the output of the approximated interpolation system, that is, $\widetilde{x}_L := \phi_L(\uparrow L)\psi x$. Then there exist a real number $C > 0$ which depends only on ϕ and L such that

$$\frac{\|z^{-dL}x_L - \widetilde{x}_L\|_2}{\|x\|_2} \leq C\|z^{-d} - \psi(z)\phi(z)\|_\infty. \quad (5)$$

Proof. Let ψ_I be the ideal filter which satisfies $\psi_I * \phi = I$. Then, by (1), we have $\psi_I x = c$, and

$$\begin{aligned} z^{-dL}x_L - \widetilde{x}_L &= z^{-dL}\phi_L(\uparrow L)\psi_I x - \phi_L(\uparrow L)\psi x \\ &= \phi_L(\uparrow L)(z^{-d} - \psi\phi)\psi_I x \\ &= \phi_L(\uparrow L)(z^{-d} - \psi\phi)c. \end{aligned}$$

Since $\{\phi(\cdot - k)\}_{k=0}^{\infty}$ is a Riesz basis [3], there exists a real number $K > 0$ which depends on ϕ and is independent of c and x such that

$$\|c\|_2 \leq K\|x\|_2.$$

Finally, we have

$$\begin{aligned} \|z^{-dL}x_L - \widetilde{x}_L\|_2 &= \|\phi_L(\uparrow L)(z^{-d} - \psi\phi)c\|_2 \\ &\leq \|\phi_L(\uparrow L)\|_\infty \|z^{-d} - \psi\phi\|_\infty \|c\|_2 \\ &\leq \|\phi_L(\uparrow L)\|_\infty \|z^{-d} - \psi\phi\|_\infty K\|x\|_2 \\ &= C\|z^{-d} - \psi\phi\|_\infty \|x\|_2, \end{aligned}$$

where $C := K\|\phi_L(\uparrow L)\|_\infty$. □

By this proposition, we conclude that if the H^∞ norm of the error system $z^{-d} - \phi(z)\psi(z)$ is adequately small, the SNR of the interpolater can be large, and hence H^∞ optimization provides a good approximation of the ideal (i.e., non-causal) spline interpolation.

We next show a relation between the performance and the reconstruction delay d.

Proposition 3: Let $\mathcal{S}$ be the set of all stable and causal IIR filters. Then, as $d \to \infty$, we have

$$\min_{\psi\in\mathcal{S}} \frac{\|z^{-dL}x_L - \widetilde{x}_L\|_2}{\|x\|_2} \to 0. \quad (6)$$

Proof. Let $J_{\mathrm{opt}}(d)$ be the minimum value of (4) when the reconstructin delay is d. Then we have [2]

$$\lim_{d\to\infty} J_{\mathrm{opt}}(d) = 0.$$

By this and Proposition 2, we have (6). □

5. CAUSAL EXPONENTIAL SPLINE INTERPOLATION

In the case of exponential splines, the function $\phi(t)$ is given by

$$\phi(t) = (\beta_{\alpha_1} * \cdots * \beta_{\alpha_N})(t) =: \beta_{\boldsymbol{\alpha}}(t),$$

where

$$\beta_{\alpha_n}(t) := \begin{cases} e^{\alpha_n t}, & 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

and $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_N)$. Then, the spline interpolation is realized as the system shown in Fig. 2, where $\phi_L(z) = H_L(z)B_L(z)$,

$$H_L(z) := \frac{1}{L^{N-1}} \prod_{n=1}^{N} \left(\sum_{k=0}^{L-1} e^{\alpha_n k} z^{-k} \right),$$

$$B_L(z) := \sum_{k=0}^{N} \beta_{\boldsymbol{\alpha}/L}(k) z^{-k},$$

and $\psi(z)$ is a filter which satisfies (3) with

$$\phi(z) = \sum_{k=0}^{N} \beta_{\boldsymbol{\alpha}}(k) z^{-k}.$$

Since the filters $H_L(z)$ and $B_L(z)$ are both FIR filters, the filter $\phi_L(z)$ is a stable transfer function. Therefore, the design problem of causal exponential spline interpolation is formulated in the same way as the case of polynomial splines mentioned in the previous section.

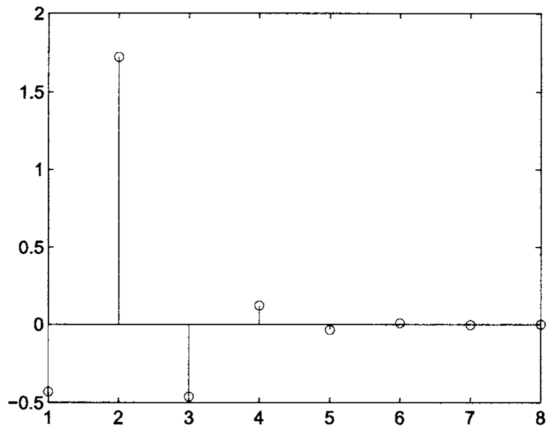

Fig. 5 Impulse response of the H^∞ optimal filter ψ_{opt}

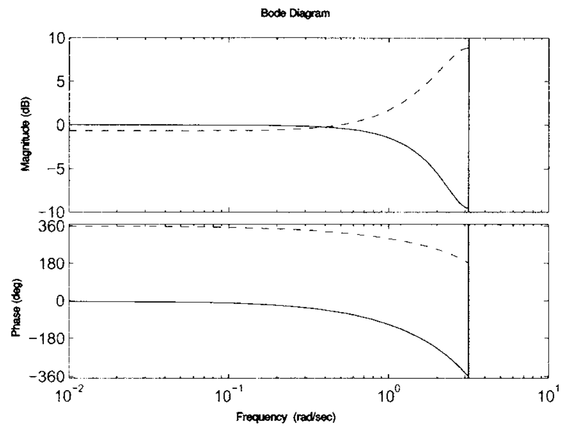

Fig. 6 Bode plot of $\phi(z)$ (solid) and $\psi_{\text{opt}}(z)$ (dash)

6. DESIGN EXAMPLE

We here present a design example of causal spline interpolation. We consider the spline of order $N = 3$ (cubic spline),

$$\phi(t) = (\beta^0 * \beta^0 * \beta^0 * \beta^0)(t),$$
$$\beta^0(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

We take the reconstruction delay $d = 3$ and design the filter ψ by the proposed H^∞ optimization. Fig. 5 shows the impulse response of the H^∞ optimal filter ψ_{opt}. Fig. 6 shows the bode plot of the original filter $\phi(z)$ and the inverse filter $\psi_{\text{opt}}(z)$. We can see that the filter $\psi(z)$ is a good approximate inverse of the original filter $\phi_{\text{opt}}(z)$ except for the phase response. The difference of the phase response is caused by the time delay z^{-d}.

Fig. 5 shows that the impulse response of $\psi_{\text{opt}}(z)$ is almost zero when $n > 5$. This implies that we need only a few taps for the filter $\psi_{\text{opt}}(z)$. Therefore, we next design an FIR filter by H^∞ optimization which is described by an LMI (see 3.3). The order of the FIR filter is $N = 10$. The optimal filter ψ_{optFIR} is shown in Table 1. The optimal value $J(\psi_{\text{opt}})$ by the IIR filter $\psi_{\text{opt}}(z)$, and $J(\psi_{\text{optFIR}})$ by the FIR filter ψ_{optFIR} are

Table 1 H^∞ optimal FIR filter

k	a_k
0	-0.43046459925698
1	1.72304266778654
2	-0.46168830495321
3	0.12370904806697
4	-0.03314771849838
5	0.00888179946408
6	-0.00237947111085
7	0.00063605079364
8	-0.00016424370228

$$J(\psi_{\text{opt}}) = 0.0720377,$$
$$J(\psi_{\text{optFIR}}) = 0.0721691.$$

It follows that the FIR filter of order $N = 8$ is almost H^∞ optimal.

7. CONCLUSION

In this article, the design of causal interpolation with polynomial and exponential splines has been proposed. The design is formulated as H^∞ optimization, which can be effectively solved by MATLAB. A further direction of this study will be to design interpolation systems with sampled-data H^∞ optimization.

REFERENCES

[1] G. Balas, R. Chiang, A. Packard, and M. Safonov, *Robust Control Toolbox Version 3*, The MathWorks, 2005.

[2] T. Chen and B. A. Francis, Design of multirate filter banks by $\mathcal{H}_\infty$ optimization, *IEEE Trans. Signal Processing*, **Vol. 43, No. 12**, pp. 2822–2830, 1995.

[3] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, 1996.

[4] M. Unser, A. Aldroubi, and M. Eden, B-Spline signal processing: Part-I — Theory, *IEEE Trans. Signal Processing,* **Vol. 41, No. 2**, pp. 821–833, 1993.

[5] M. Unser, A. Aldroubi, and M. Eden, B-Spline signal processing: Part-II — Efficient design and applications, *IEEE Trans. Signal Processing,* **Vol. 41, No. 2**, pp. 834–848, 1993.

[6] M. Unser and T. Blu, Cardinal exponential splines: Part I — Theory and filtering algorithms, *IEEE Trans. Signal Processing,* **Vol. 53, No. 4**, pp. 1425–1438, 2005.

[7] Y. Yamamoto, B. D. O. Anderson, M. Nagahara, and Y. Koyanagi, Optimizing FIR approximation for discrete-time IIR filters, *IEEE Signal Processing Letters*, **Vol. 10, No. 9**, pp. 273–276, 2003.