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Interpolation of Nonuniformly Decimated Signals via Sampled-data H^∞ Optimization

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Abstract: In this paper, we consider signal interpolation of discrete-time signals which are decimated nonuniformly. A conventional interpolation method is based on the sampling theorem, and the resulting system consists of an ideal filter with complex coefficients. On the other hand we adopt sampled-data H^∞ optimization, which can take account of intersample behavior, and the optimal filter with real coefficients is obtained. An example shows the effectiveness of our method. By examples, we also show that there is the optimal decimation pattern.

Keywords: nonuniform decimation, interpolation, sampled-data control

1. INTRODUCTION

Interpolation is a fundamental operation in digital signal processing, and has many applications such as signal reconstruction, signal compression/expansion, and resizing/rotating digital images, see [11], [3]. If digital data to be interpolated are located uniformly on the time axis, the *uniform* interpolation is executed by an expander and a digital filter (called an interpolation filter) [11], which is conventionally designed via the sampling theorem.

Periodic nonuniform interpolation (or decimation) also plays an important role in signal processing, such as signal compression by nonuniform filterbanks [8], super-resolution image processing [9], and time-interleaved AD converters [10]. The design has been studied by many researchers [12], [8], [13], [4], [5], in which the design methods are based on the generalized sampling theorem. The optimal filter (or the perfect reconstruction filter) is an ideal lowpass filter with complex coefficients [12], [11]. Since the ideal filter cannot be realized, approximation methods are also proposed, see in particular [12], [13].

On the other hand, real signals such as audio signal (esp. orchestral music) breaks the band-limiting assumption in the sampling theorem, that is, they have some frequency components beyond the Nyquist frequency. In view of this, we have to take account of the *whole* frequency range in designing interpolation systems. *Sampled-data H^∞ optimization* [1], [7] is very adequate for this purpose.

In this article, we first define nonuniform decimation/interpolation. This definition includes the block decimation introduced in [8]. Then we formulate the interpolation problem as a sampled-data H^∞ optimization. The optimal filter is given by a periodic system, which can be realized by a multirate filterbank. Design examples show the effectiveness of our method. Moreover, we consider by examples what is the optimal decimation pattern. We show that although the decimation rate is the same, the optimal value can differ if the pattern differs. That is, the performance depends on the decimation pattern. This property can be used in designing signal compression.

2. NONUNIFORM DECIMATION AND INTERPOLATION

Consider a discrete-time signal $x := \{x_0, x_1, x_2, \dots\}$ shown in Fig. 1. Then nonuniform decimation by $\boldsymbol{M} := [1, 1, 0]$ (we call this a *decimation pattern*) is defined as follows (see Fig. 2).

$$(\downarrow \boldsymbol{M})x := \{x_0, x_1, x_3, x_4, x_6, \dots\}. \tag{1}$$

That is, we first divide the time axis into segments of length three (the number of the elements of $\boldsymbol{M}$), then, in each segment, retain the samples corresponding to 1 in $\boldsymbol{M}$ and discard the one corresponding to 0. This decimation includes so-called block decimation, in which the first R_1 samples of each segment of R_2 samples are retained while the rest are discarded [8]. By using our notation, the block decimation $R_2 : R_1$ is represented as $\downarrow \boldsymbol{M}$ with

$$\boldsymbol{M} = [\underbrace{1, \dots, 1}_{R_1}, \underbrace{0, \dots, 0}_{R_2 - R_1}].$$

Then we consider interpolation. First, we define the nonuniform expander $\uparrow \boldsymbol{M}$ with $\boldsymbol{M} = [1, 1, 0]$ by

$$(\uparrow \boldsymbol{M})x := \{x_0, x_1, 0, x_2, x_3, 0, x_4, \dots\}.$$

That is, we first divide the time axis into segment of length two (the number of the elements 1 of $\boldsymbol{M}$), then insert 0 into the portion corresponding to 0 in $\boldsymbol{M}$. Applying this to the decimated sequence (1), we have

$$v := (\uparrow \boldsymbol{M})(\downarrow \boldsymbol{M})x = \{x_0, x_1, 0, x_3, x_4, 0, x_6, \dots\}.$$

Then, the interpolation is completed by filtering v by a digital filter $\mathcal{K}$ (see Fig. 3 (a)).

3. DESIGN OF INTERPOLATION FILTER

3.1 Design Problem

In this section, we consider a general case where the decimation pattern is defined by

$$\boldsymbol{M} := [b_1, b_2, \dots, b_M], \quad b_i \in \{0, 1\}.$$

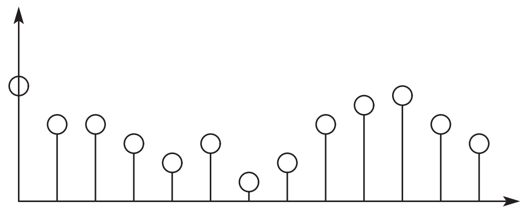


Fig. 1 Discrete-time signal x

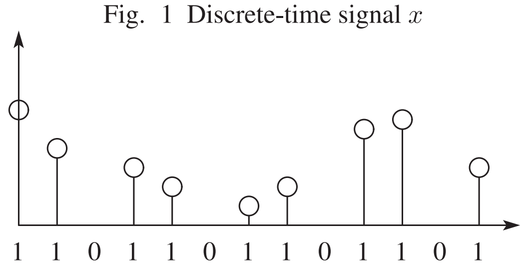


Fig. 2 Nonuniform decimation ($\boldsymbol{M} = [1, 1, 0]$)

Let $i_1, i_2, \ldots, i_N$ ($i_1 < i_2 < \ldots < i_N$) be the indices of b_i's such that $b_{i_1} = \cdots = b_{i_N} = 1$ (N is the number of ones in $\boldsymbol{M}$). Then the interpolation process $\mathcal{K}(\uparrow \boldsymbol{M})(\downarrow \boldsymbol{M})$ is periodically time-varying, more precisely, it is (N, M)-periodic [6] or (N, M)-shift-invariant [2]. By this fact, the multirate system can be equivalently represented as a time-invariant system via discrete-time lifting [1], which is also called polyphase decomposition [11]. Let $\mathbf{L}_M$ be the discrete-time lifting operator, that is,

$$\mathbf{L}_M : \{x_0, x_1, \ldots\} \mapsto \left\{ \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{M-1} \end{bmatrix}, \begin{bmatrix} x_M \\ x_{M+1} \\ \vdots \\ x_{2M-1} \end{bmatrix}, \ldots \right\}.$$

By this, the interpolation process is equivalently represented as

$$\mathcal{K}(\uparrow \boldsymbol{M})(\downarrow \boldsymbol{M}) = \mathbf{L}_N^{-1} \widetilde{\mathcal{K}} E \mathbf{L}_M, \quad (2)$$

$$\widetilde{\mathcal{K}} := \mathbf{L}_N \mathcal{K} \mathbf{L}_N^{-1}, \quad (3)$$

where $E = [E_{ij}]$ is an $N \times M$ matrix whose elements are defined as follows:

$$E_{ij} = \begin{cases} 1, & (i, j) = (1, i_1), (2, i_2), \ldots, (N, i_N), \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $\boldsymbol{M} = [1, 1, 0]$ ($M = 3$, $N = 2$, $i_1 = 1$, $i_2 = 2$) then

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the lifted filter $\widetilde{\mathcal{K}}$ is a time-invariant system with N-inputs and N-outputs.

To design the interpolation filter $\mathcal{K}$ (or $\widetilde{\mathcal{K}}$), consider the error system shown in Fig. 4. In this figure, F is a linear time-invariant continuous-time system whose transfer function is finite-dimensional and strictly proper, which is a model of the original analog signal. The block $\mathcal{S}_h$ represents the ideal sampler with sampling period h, and

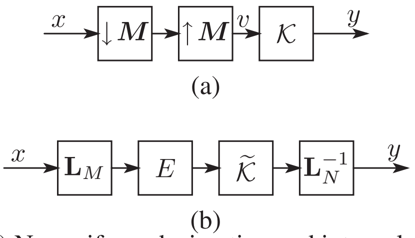


Fig. 3 (a) Nonuniform decimation and interpolation, (b) lifted system

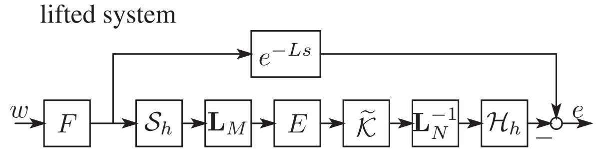


Fig. 4 Error system

$\mathcal{H}_h$ the zero-order hold with the same sampling period. The delay e^{-Ls} is a design parameter which control the reconstruction delay and the performance Then our problem is formulated as a sampled-data H^∞ optimization.

Problem 1: *Find the optimal filter $\widetilde{\mathcal{K}}$ which minimizes the H^∞ norm (the L^2-induced norm) of the error system T_{ew} from the continuous-time signal w to the error e. The L^2-induced norm of T_{ew} is defined by*

$$\|T_{ew}\| := \sup_{\substack{w \in L^2 \\ w \neq 0}} \frac{\|T_{ew} w\|_2}{\|w\|_2}. \quad (4)$$

Note that the norm (4) equals the H^∞ norm of the sampled-data system T_{ew} [1]. The problem is therefore called a *sampled-data H^∞ optimization problem*.

3.2 Filter design and implementation

The problem formulated above is a standard sampled-data signal reconstruction problem. The H^∞ optimal filter $\widetilde{\mathcal{K}}$ can be obtained by the fast-sampling method, see [7].

The filter $\mathcal{K}$ is obtained by the inverse discrete-time lifting of (3), that is

$$\mathcal{K} = \mathbf{L}_N^{-1} \widetilde{\mathcal{K}} \mathbf{L}_N.$$

Then the interpolation system is $\mathcal{K}(\uparrow \boldsymbol{M})$, see Fig.3 (a) and the equation (2).

There is however another simpler way to implement the interpolation system, by using a multirate filterbank, see Fig. 5. In this filterbank, $F_{i_1}(z)$, $F_{i_2}(z)$, ..., $F_{i_N}(z)$ are obtained by the following equation:

$$\begin{bmatrix} F_{i_1}(z) \\ F_{i_2}(z) \\ \vdots \\ F_{i_N}(z) \end{bmatrix} = \widetilde{\mathcal{K}}(z^N)^\top \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N+1} \end{bmatrix}.$$

The uniform decimator $\downarrow M$ and expander $\uparrow M$ where M is a positive integer is defined by

$$y = (\downarrow M)x, \quad y_n = x_{Mn}, \quad n = 0, 1, 2, \ldots$$

$$w = (\uparrow M)v, \quad w_n = \begin{cases} x_{n/M}, & \text{if } n = 0, M, 2M, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

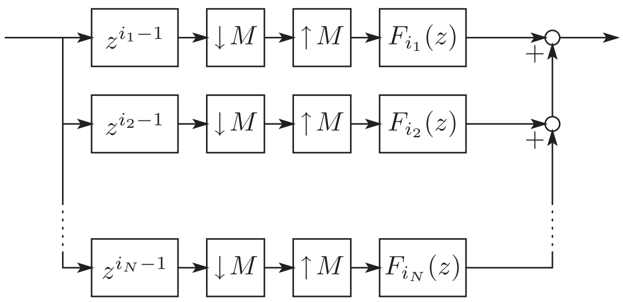


Fig. 5 Nonuniform filterbank

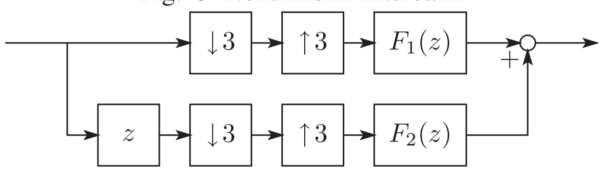


Fig. 6 Nonuniform filterbank ($\boldsymbol{M} = [1, 1, 0]$)

By using our definition of nonuniform decimator and expander, the uniform ones are given by

$$\downarrow M = \downarrow \boldsymbol{M}, \quad \uparrow M = \uparrow \boldsymbol{M}, \quad \boldsymbol{M} = [1, \underbrace{0, 0, \ldots, 0}_{M-1}].$$

Fig. 6 shows an example of a nonuniform filterbank when $\boldsymbol{M} = [1, 1, 0]$.

4. DESIGN EXAMPLES

In this section, we show design examples.

4.1 Optimal filter design

Here we design the optimal filter $\mathcal{K}$ (or $F_{i_1}, \ldots F_{i_N}$ in Fig. 5). The design parameters are as follows: the decimation vector $\boldsymbol{M} = [1, 1, 0]$, the sampling period $h = 1$, the time delay $L = 12$. The analog characteristic of the continuous-time input signals is modeled by

$$F(s) = \frac{1}{10s + 1}. \tag{5}$$

For comparison, we adopt the method of Hilbert transformer [12], [11] as a conventional one. Note that this method is based on the sampling theorem, assuming that the original analog signal is fully band-limited up to the frequency $\omega = 2\pi/3$ ($2/3$ of the Nyquist frequency π). Note also that the conventional filter requires very large delay ($L = 61.5$). Fig. 7 shows the frequency responses of the error system T_{ew} shown in Fig. 4. The conventional interpolation shows a large error in high frequency, while the sampled-data H^∞ optimal interpolation shows a flat response. To illustrate the difference between these responses, we simulate interpolation of a rectangular wave. Fig. 8 shows the time response. The conventional interpolation causes large ripples, while our interpolation shows a better response. This is because the rectangular wave has high frequency components around the edges, and our interpolation takes account of such frequency components.

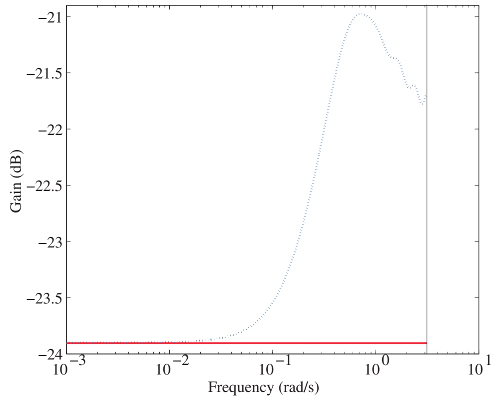


Fig. 7 Frequency response: proposed (solid) and conventional (dots)

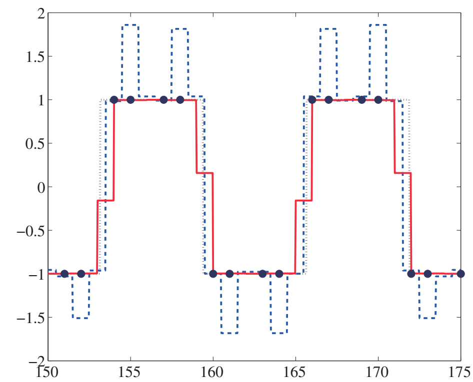


Fig. 8 Time response: proposed (solid), conventional (dash), and input signal(dots)

4.2 Decimation pattern analysis

Consider $M = 3$ and $N = 2$. Then there are three patterns of decimation: $\boldsymbol{M}_1 = [1, 1, 0]$, $\boldsymbol{M}_2 = [1, 0, 1]$, and $\boldsymbol{M}_3 = [0, 1, 1]$. These are essentially the same except for delays, that is,

$$z^{-1}(\uparrow \boldsymbol{M}_2)(\downarrow \boldsymbol{M}_2) = (\uparrow \boldsymbol{M}_1)(\downarrow \boldsymbol{M}_1)z^{-1},$$
$$(\uparrow \boldsymbol{M}_3)(\downarrow \boldsymbol{M}_3)z^{-1} = z^{-1}(\uparrow \boldsymbol{M}_1)(\downarrow \boldsymbol{M}_1).$$

The optimal values of our H^∞ optimization for these decimations are therefore the same. However, when $M = 4$ and $N = 2$, there can be difference. In this case, the essential patterns are $\boldsymbol{M} = [1, 1, 0, 0]$ or $\boldsymbol{M} = [1, 0, 1, 0]$. Set the design parameters $F(s)$ as (5), $h = 1$, and $L = M$ (the length of the segment). Table 1 shows the optimal values $\gamma = \min \|T_{ew}\|$ (see (4)). This result shows that the pattern $\boldsymbol{M} = [1, 0, 1, 0]$ (or $\boldsymbol{M} = [0, 1, 0, 1]$) is the better, which is equals to the uniform decimation $\downarrow 2$.

We then design when the segment length $M = 5$. Table 2 shows the result. By this, the optimal value γ depends on the position of the zeros in $\boldsymbol{M}$, not depends on the number of the ones in $\boldsymbol{M}$. For example, although the pattern (F) retains more samples than the pattern (E), the optimal values are the same. Table 3 shows the optimal γ when $M = 7$ and $N = 4$. In this case, there are 5 the

Table 1 Optimal value γ for $M = 4$ and $N = 2$

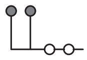
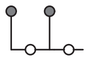
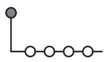
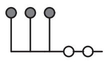
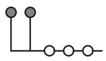
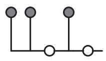
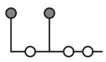
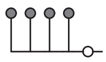
Decimation Pattern	γ
 $\boldsymbol{M} = [1100], [1001], [0110], [0011]$	0.2293
 $\boldsymbol{M} = [1010], [0101]$	0.1529

Table 2 Optimal value γ for $M = 5$ and $N = 1, 2, 3, 4$

Pattern	γ	Pattern	γ
 (A)	0.3813	 (D)	0.2303
 (B)	0.3062	 (E)	0.1536
 (C)	0.2303	 (F)	0.1536

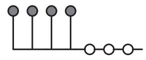
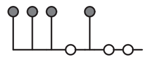
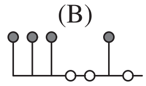
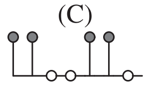
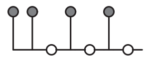
essential patterns (A) to (E). We can see that γ depends on the maximal number of the *consecutive zeros* in $\boldsymbol{M}$ (we here call this the consecutive number).

The results shows that the reconstruction performance depends on the consecutive number and not on the number of retained samples. By this observation, we can conjecture that *the optimal decimation pattern $\boldsymbol{M}$ is the pattern in which the zeros are least consecutive*. In other words, *most uniformly distributed pattern is the best*. In view of this, the block decimation introduced in [8] cannot be optimal.

5. CONCLUSION

We have proposed an interpolation method of nonuniform decimation via sampled-data H^∞ optimization. By a design example we have shown the effectiveness of our method. We have also suggested by examples that there can be the optimal decimation pattern.

Table 3 Optimal value γ for $M = 7$, $N = 4$

Decimation Pattern	γ
 (A)	0.3089
 (B)  (C)  (D)	0.2320
 (E)	0.1547

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