DIGITAL FILTER DESIGN WITH OPTIMAL ANALOG PERFORMANCE

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Abstract

This paper proposes a new digital filter design methodology, based on sampled-data control theory. In contrast to the conventional filter designs where the methods are mostly based on frequency domain approximation techniques, the proposed method makes use of the sampled-data H^∞ control theory which has been quite successful in recent years in the control literature. The novel feature here is that the proposed method can optimize the analog-domain performance over all frequency ranges, thereby guaranteeing a desirable performance without breaking the design problem into several different steps, such as linear phase characteristic, optimal attenuation level design, etc. A design example is presented to show the advantages of the present method.

1. Introduction

Digital filter design is an art of approximation which takes many different specifications into account: linear phase shift property, smooth pass-band transmission, high attenuation level in the stop band, desirable transition band characteristic, etc. Many guiding quantities are there to help the designer [8, 13, 9].

The design is now performed mostly in the discrete-time domain. To capture the continuous-time performance, the notion of aliasing is utilized and deviation from the ideal filter has to be discussed. To bypass the problem of the Gibbs phenomenon in the frequency domain windows are often effective.

One may however note that, in many applications, the performance we wish to optimize is still in the analog domain: speech/audio is one example; visual images are another. While one may start with the digitized data in which case an analog-domain performance cannot be adequately discussed, there are many other cases where we can discuss the basic characteristics of the original analog data. For example, in audio recordings, we have a fairly good idea on how the frequency characteristics are for recorded signals. Recovering such signals optimally in the sense of analog performance is clearly an important issue.

This paper proposes a new digital filter design methodology, based on sampled-data control theory. In contrast to the conventional filter designs, this design method does not rely on an approximation techniques (e.g., frequency sampling). Instead, it gives rise to an optimal transfer operator, where the performance is measured by the H^∞ norm. In contrast to the more popular H^2 norm, which measures only the mean-square performance of the frequency response, the H^∞ norm measures the *supremum* of the gain of the frequency response. By multiplying a suitable frequency weighting function, we can control the attenuation level fairly precisely. The price is that this norm does not make the underlying signal space a Hilbert space; H^∞ is only a Banach space. Hence the standard technique for approximation such as the projection theorem cannot be used, and optimization in this space is indeed good deal more difficult than that in H^2 which is a Hilbert space. However, it is more natural and adequate for many applications as a performance measure and this explains the recent boost of applications of H^∞ control after this problem was solved in a satisfactory form (see, e.g., [4]).

This development is further generalized to the sampled-data context where measurement and control actions occur in discrete time. The theory for sampled-data H^∞ control is now fairly complete; the important feature here is that sampled-data control optimizes continuous-time (analog) H^∞ performance, while maintaining discrete-time control actions [2]. There are also remarkable differences between sampled-data and discrete-time designs.

Such a development provides an optimal platform for designing digital filters. An attempt is made in [3] for an multirate filter bank design problem. Other approaches have also been made, e.g., [7, 10, 11]. However, a low-pass filter design problem with optimal analog performance has not been formulated or solved there.

This paper considers the design of an optimal low-pass filter design when one employs an upsampler. The objective is to reconstruct the original signal in this situation. Usually this problem is dealt with under the assumption that

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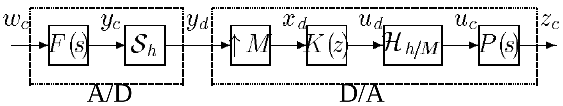

Figure 1: Multirate Signal Reconstruction

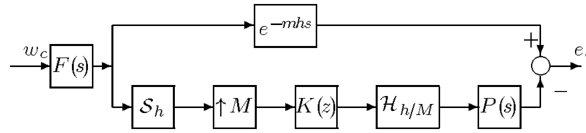

Figure 2: Signal reconstruction error system

the original signal be band-limited; in practice, however, no signals are entirely band-limited. Instead, we may often know an approximate frequency characteristic that the original signals obey, and it is these original analog signals we wish to reconstruct optimally.

We thus formulate a digital signal reconstruction problem under the following assumptions:

- the orignal analog signal is subject to a certain frequency characteristic, but not fully band-limited;
- the digital signal can be upsampled to employ a faster hold device;
- the overall analog H^∞ performance must be optimized.

This may also be regarded as an optimal D/A converter design. We will see that performance improvement is possible over a conventional low-pass filter. Even though we do not explicitly place the constraint on linear-phase property, it is interesting to note that the obtained filter is very close to linear phase up to the Nyquist frequency, due to the H^∞ performance requirement.

2. Problem Formulation

Consider the block diagram Figure 1. The incoming signal w_c first goes through an anti-aliasing filter $F(s)$ and the filtered signal y_c becomes nearly (but not entirely) band-limited. $F(s)$ governs the frequency-domain characteristic of the analog signal y_c. This signal is then sampled by $\mathcal{S}_h$ to become a discrete-time signal y_d with sampling period h. This signal is usually stored or transmitted with some media (e.g., CD) or a channel.

To restore y_c we first upsample the discrete-time signal y_d by factor M:

$$\uparrow M : y_d \mapsto x_d : x_d[k] = \begin{cases} y_d[l], & k = Ml,\ l = 0, 1, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

The signal then becomes another discrete-time signal x_d with sampling period h/M. The discrete-time signal x_d is then processed by a digital filter $K(z)$, becomes a continuous-time signal u_c by going through the 0-order hold $\mathcal{H}_{h/M}$ (that works in sampling period h/M), and then becomes the final signal by passing through an analog filter $P(s)$. An advantage here is that one can use a fast hold device $\mathcal{H}_{h/M}$ thereby making more precise signal restoration possible. The objective here is to design the digital filter $K(z)$ for given $F(s)$, M and $P(s)$.

Figure 2 shows the block diagram for the error system for the design. The delay in the upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. Let T_{ew} denotes the input/output operator from w_c to $e_c(t) := z_c(t) - u_c(t - mh)$. Our design objective is as follows:

Problem 1 *Given stable $F(s)$ and $P(s)$ and an attenuation level $\gamma > 0$, find a digital filter $K(z)$ such that*

$$\|T_{ew}\|_\infty := \sup_{w_c \in L^2[0,\infty)} \frac{\|T_{ew} w_c\|_2}{\|w_c\|_2} < \gamma, \qquad (1)$$

where $\|\cdot\|_2$ denotes the L^2 norm. An advantage of defining this problem is that we do not need any extra design constraint (such as the linear phase property). Such issues are incorporated into the design diagram Fig. 2.

3. Reduction to A Finite-Dimensional Problem

A difficulty in Problem 1 is that it involves a continuous time-delay, and hence it is an infinite-dimensional problem. Another difficulty is that it contains the upsampler $\uparrow M$, so that it makes the overall system time-varying.

Following the method of [7, 10], however, we can reduce this problem to a finite-dimensional single-rate problem:

Theorem 1 *There exist (finite-dimensional) discrete-time systems $G_{d11}(z)$, $G_{d12}(z)$ and $G_{d21}(z)$ such that (1) is equivalent to*

$$\|z^{-m} G_{d11}(z) - G_{d12}(z)\widetilde{K}(z)G_{d21}(z)\|_\infty < \gamma, \qquad (2)$$

where $\widetilde{K}(z)$ is the blocking system of $K(z)$.

Proof We first reduce the problem to a single-rate problem. Define the blocking operator $\mathbf{L}_M$ and its inverse $\mathbf{L}_M^{-1}$ by

$$\begin{aligned} \mathbf{L}_M &:= (\downarrow M) \begin{bmatrix} 1 & z & \cdots & z^{M-1} \end{bmatrix}^T \\ \mathbf{L}_M^{-1} &:= \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M+1} \end{bmatrix} (\uparrow M), \end{aligned}$$

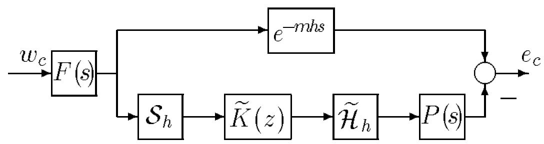

Figure 3: Reduced single-rate problem

where $\downarrow M$ denotes the downsampler

$$\downarrow M : x_d \mapsto y_d : y_d[k] = x_d[Mk].$$

Then $K(z)(\uparrow M)$ can be rewritten as

$$\begin{aligned} K(z)(\uparrow M) &= \mathbf{L}_M^{-1}\widetilde{K}(z) \\ \widetilde{K}(z) &:= \mathbf{L}_M K(z)\mathbf{L}_M^{-1} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T. \end{aligned}$$

$\widetilde{K}(z)$ is an LTI, single-input/M-output system that satisfies

$$K(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M+1} \end{bmatrix} \widetilde{K}(z^M).$$

Using the generalized hold $\widetilde{\mathcal{H}}_h$ defined by

$$\begin{aligned} \widetilde{\mathcal{H}}_h &: l^2 \ni v \mapsto u \in L^2, \ \ u(kh+\theta) = \mathbf{H}(\theta)v[k] \\ & \quad \theta \in [0,h), \ \ k = 0,1,2,\ldots \\ \mathbf{H}(\theta) &:= \begin{cases} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}, & \theta \in [0, h/M) \\ \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix}, & \theta \in [h/M, 2h/M) \\ \ldots & \\ \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, & \theta \in [(M-1)h/M, h) \end{cases} \end{aligned}$$

we obtain the identity

$$\mathcal{H}_{h/M}\mathbf{L}_M^{-1} = \widetilde{\mathcal{H}}_h.$$

This yields

$$\mathcal{H}_{h/M}K(z)(\uparrow M)\mathcal{S}_h = \widetilde{\mathcal{H}}_h\widetilde{K}(z)\mathcal{S}_h.$$

Hence Figure 2 is equivalent to Figure 3. We can then invoke the technique of [7] to reduce this to a finite-dimensional design problem (2). □

4. Approximation via Fast Sample/Hold

While the procedure above reduces Problem 1 to a finite-dimensional H^∞ problem, it is in general not numerically suitable for actual computation; the formulas are quite involved, and not so numerically tractable. It is often more convenient to resort to an approximation method. We employ the fast sample/hold approximation [2, 10]. This method approximates continuouos-time inputs and outputs via a sampler and hold that operate in the period h/M. The convergence of such an approximation is guaranteed in [12].

By this method, our design problem (1) is approximated as

$$\|z^{-m}G_{dN11}(z) + G_{dN12}(z)\widetilde{K}(z)G_{dN21}(z)\|_\infty < \gamma,$$

where G_{dN11}, G_{dN12} and G_{dN21} are discrete-time systems obtained by the technique in [2, 10]. Once the problem has been reduced to such a problem, it can be solved by a control design toolbox such as those given by MATLAB (see, e.g., [1]).

5. A Design Example

We present a design example for

$$F(s) = \frac{1}{(0.70223s+1)(7.0223s+1)}, \quad P(s) = 1$$

with $h = 1$, $m = 2$ and upsampling factor $M = 2$. (In commercial CD players, M is usually $8 \sim 32$.) An approximate design is executed here for $N = M \times 4 = 8$. For comparison, we compare it with the Johnston filter[6] of order 31, which is often used in commercial applications. For the CD format, the above frequency response corresponds to the decay of -20dB/dec beyond 1kHz, and -40dB/dec beyond 10kHz. In many such applications, we often have such a frequency characteristic for original analog signals.

The computed (via MATLAB [1]) (sub) optimal filter $K_{SD}(z)$ is IIR of order 7 with the filter coefficients ($\mathbf{b}$: numerator, $\mathbf{a}$: denominator) of the transfer function:

$$\mathbf{b} = \begin{bmatrix} -0.053136 \\ -0.081530 \\ 0.252018 \\ 0.809684 \\ 0.917880 \\ 0.512434 \\ 0.134899 \\ 0.010296 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 1.000000 \\ 0 \\ 0.255479 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The impulse response of this filter becomes almost zero after 12 $\sim$ 14 steps, so that it can be well approximated by an FIR filter of 14 taps.

Figure 4 shows the gain characteristics of these filters. The Johnston filter shows a very sharp decay beyond the cutoff frequency (π [rad/sec]) and the sampled-data design shows a rather slow decay. If the original signals are ideally band-limited, then a sharp decay such as that of the Johnston filter would be desirable. But if the original signal is not fully band-limited, but obeys the frequency characteristic as specified by $F(s)$, such a sharp cut in the transiiton band may not be optimal.

In fact, the reconstruction error characteristic in Figure 5 shows that the performance of $K_{SD}(z)$ is comparable to that of the 32 tap Johnston filter. Furthermore, we

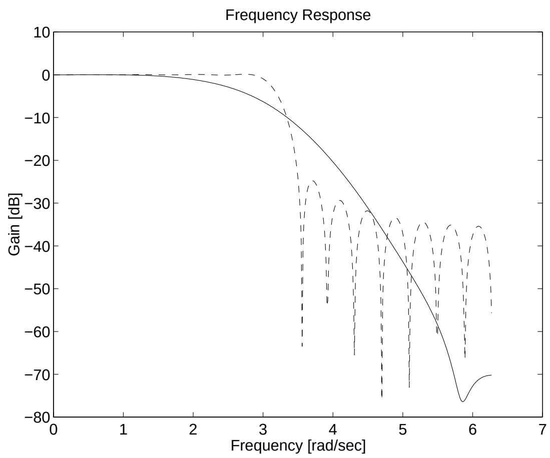

Figure 4: Frequency response of filter: $K_{SD}(z)$ (solid) and $K_J(z)$ (dash)

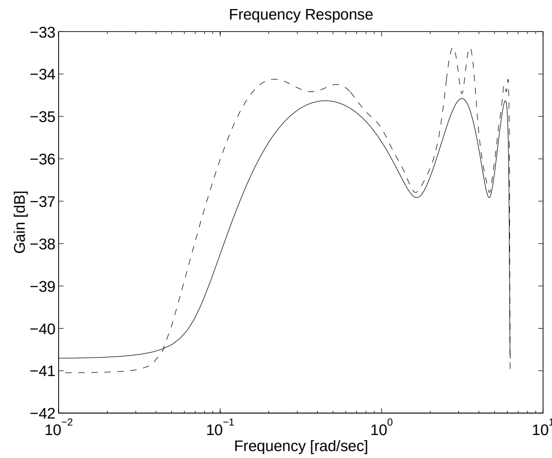

Figure 5: Frequency response of error system T_{ew} : sampled-data H^{∞} synthesis (solid) and Johnston filter (dash)

have made sure (frequency response plot omitted) that the sampled-data design gives 6-10 dB improvement over the Johnston filter when we increase the upsampling factor to $M = 4$.

Furthermore, in the time-domain, the sampled-data design shows a clear advantage over the Johnston filter. Let us see the time responses against rectangular waves in Figures 6, 7:

While the Johnston filter exhibits a very typical Gibbs phenomenon, the one by $K_{SD}(z)$ has much less peak around the edge. We also note that $K_{SD}(z)$ is nearly linear phase, as shown in Figure 8. The Gibbs phenomenon in the conventional design is simply an outcome of the very sharp cut beyond the cut-off frequency, since it corresponds to the truncation of a Fourier series with finitely many terms.

To see the effect, the obtained filter is applied to downsampled (by factor 2) CD signals, and less distortion has been heard compared to that processed by the Johnston filter.

6. Concluding Remarks

We have presented a new method of designing a digital filter in multirate signal reconstruction problem. An advantage here is that an analog optimal performance can be obtained, and this can be advantageous in audio signal reconstruction. Another advantage is that the design can be done in essentially one process, using MATLAB routines (e.g., [1]), once the problem is formulated.

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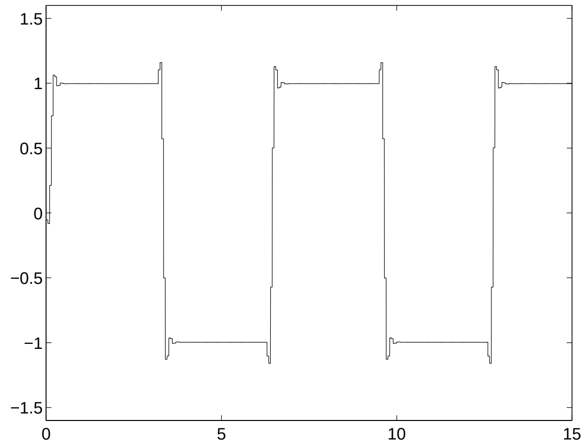

Figure 6: Time response (sampled-data design)

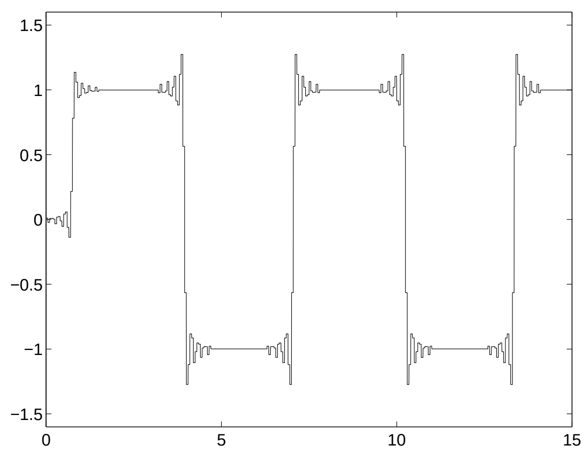

Figure 7: Time response (Johnston filter)

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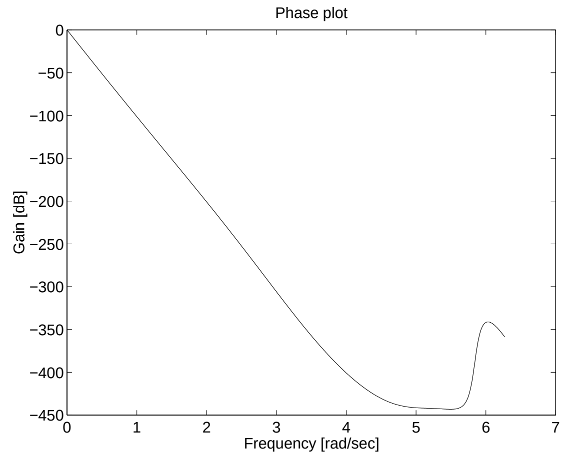

Figure 8: Phase plot of K_{SD}

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