Design of $\Delta\Sigma$ Converters via Sampled-Data H^∞ Optimization

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Abstract

In this paper, we propose a new method for designing $\Delta\Sigma$ converters via sampled-data H^∞ optimal control. The design consists of two steps. One is that for $\Delta\Sigma$ modulators. In $\Delta\Sigma$ modulators, the accumulator $1/(z-1)$ is conventionally used in a feedback loop to attenuate quantization noise. In contrast, we give all stabilizing controllers for the modulator, and propose an H^∞ design to shape the frequency response of the system from the noise to the output. The other is a design for multirate filters in oversampling AD/DA converters. While conventional designs are executed in the discrete-time domain, we take account of the characteristic of the original analog signal by using sampled-data H^∞ optimization. Design examples are presented to show that our design is superior to conventional ones.

1 Introduction

$\Delta\Sigma$ *modulators* are widely used in high-resolution AD (Analog-to-Digital) or DA (Digital-to-Analog) converters, which are applied to measurement, digital audio processing and wireless communications (see [6, 12, 7]). In combination with oversampling technique, $\Delta\Sigma$ AD or DA converters can have high resolution despite quantizing signals by a coarse (by usual one-bit) quantizer. They reduce quantization noise by linear filters in feedback loops, which shape the frequency distribution of noise.

The analysis and design are commonly done by assuming that the quantization noise is white, and independent of the input signal. Although the assumption is not strictly valid, the method leads to a linear model. We can then adopt linear system theory, in particular, frequency domain approach. Noise shaping in the frequency domain can be executed by the established H^2 or H^∞ control, and hence such a linear model will be useful to design $\Delta\Sigma$ modulators. However, stability analysis cannot be easily applied because coarse quantizers in feedback loops are highly nonlinear. To solve this problem, we directly treat the nonlinearity by applying a Lyapunov method. By using this, we will show a sufficient condition for stability.

On the other hand, a novel approach to designing $\Delta\Sigma$ converters is proposed by Quevedo and Goodwin [8]. They design a generalized $\Delta\Sigma$ converter, called *Multi-Step-Optimal-Converter* (MSOC), by formulating the design problem as a weighted H^2 optimization, which is reduced to a constrained multi-step optimization. Motivated by this work, we propose a new design method of $\Delta\Sigma$ converters. Since AD or DA converters involve both continuous- and discrete-time signals, it is necessary for analysis and design to take the characteristics of both of them into account. For this purpose, the *sampled-data control* [1, 11] is an optimal tool. In the last few years, several studies have been made on digital signal processing via sampled-data control theory [4, 5]. Based on these studies, we propose sampled-data H^∞ optimization for designing $\Delta\Sigma$ converters. By sampled-data H^∞ optimization, we can optimize intersample response

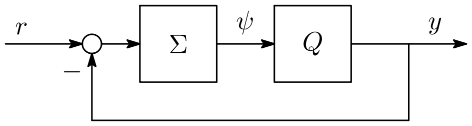


Figure 1: ΔΣ Modulator

of the signals in ΔΣ converters, while only sampled values are optimized by conventional methods. We present design examples to show that our design is superior to conventional ones.

2 ΔΣ Modulators

In this section, we study ΔΣ modulators. First, we introduce a linear model for quantization. By using this linear model, we characterize all stabilizing controllers (including the conventional controller). Then, regarding quantizers as nonlinear systems, we investigate stability of ΔΣ modulators.

2.1 Conventional modulators

Fig. 1 shows the block diagram of the conventional ΔΣ modulator. In this figure, the difference between the input r and the output y is fed back to Σ, which is conventionally an accumulator

$$\Sigma(z) = \frac{z^{-1}}{1 - z^{-1}},$$

and outputs a signal ψ.

Then the signal ψ is quantized by a quantizer Q, which is a piecewise constant function $\mathbb{R} \to \mathcal{Q}$ where $\mathcal{Q}$ is a finite subset of $\mathbb{R}$.

The quantizer Q is a nonlinear system. To make the analysis easy, we introduce a linear model for Q. Define the quantization error n, that is, $n := Q(\psi) - \psi$. Assuming that the error n is independent of the input ψ, we take the additive noise model for the quantizer as shown in Fig. 2. By using this model, we see that the input-output equation is obtained by

$$y = \frac{\Sigma}{1+\Sigma} r + \frac{1}{1+\Sigma} n = z^{-1} r + (1 - z^{-1}) n.$$

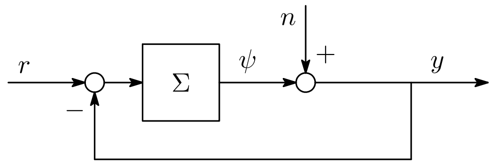


Figure 2: Linear Model for ΔΣ Modulator

Since $1/(1+\Sigma) = 1 - z^{-1}$ is high-pass, the quantization noise is reduced at low frequencies and increased at high frequencies. If the input signal r contains few high frequency components, we can separate the noise n from the output signal y by an appropriate lowpass filter.

Therefore, the accumulator Σ plays a noise-shaping role in ΔΣ modulators. In the next section, we generalize this property.

2.2 Quantization noise shaping

We here consider the linear model in Fig. 2. Let $\mathcal{S}$ denote the family of all stable, proper, real-rational transfer functions and

$$\mathcal{S}' := \{G \in \mathcal{S} : G \text{ is strictly proper}\}.$$

Then we characterize Σ for the linear system in Fig. 2.

Lemma 1. *The linearized feedback system Fig. 2 is well-posed and internally stable if and only if*

$$\Sigma \in \left\{ \frac{R}{1-R} : R \in \mathcal{S}' \right\}$$

Proof. See chapter 5 in [2]. □

This lemma gives all stabilizing feedback filters. By using the parameter $R \in \mathcal{S}'$, the input/output relation of the system Fig. 2 is given by

$$y = Rr + (1-R)n =: Rr + Hn, \qquad (1)$$

where H is the noise transfer function (NTF) to be designed. Note that the conventional first order ΔΣ modulator has $R(z) = z^{-1} \in \mathcal{S}'$.

In implementation, finite-impulse response (FIR) filters are often preferred, and hence we assume R is an FIR filter (so is the NTF H), that is,

$$R(z) = \sum_{k=1}^{N} a_k z^{-k}, \quad H(z) = 1 - \sum_{k=1}^{N} a_k z^{-k}$$

Note that $R(z)$ is always in $\mathcal{S}'$. If a desired NTF $H_{\mathrm{des}}(z)$ is given by an FIR filter, $R(z)$ is obtained by $R(z) = 1 - H_{\mathrm{des}}(z)$. Since $R(z)$ must be strictly causal, we have to restrict $H_{\mathrm{des}}(\infty) = 1$.

On the other hand, if $H_{\mathrm{des}}(z)$ is given by an IIR filter, our problem is to approximate $H_{\mathrm{des}}(z)$ by an FIR filter $H(z)$. Since desired NTFs are given by their frequency characteristics, approximation of $H(z)$ should be done in the frequency domain. Therefore, we formulate our problem as an H^∞ optimization:

Problem 1. *Given a stable transfer function $H_{\mathrm{des}}(z)$ (desired NTF) and a stable weighting function $W(z)$, find $H \in \mathcal{S}$ with $H(\infty) = 1$ which minimize $\|(H - H_{\mathrm{des}})W\|_\infty$.*

Then the optimization is reducible to a linear matrix inequality (LMI) with respect to a matrix variable and the coefficients $a_1, \ldots, a_N$ [10], and can be effectively solved by standard optimization software (e.g., MATLAB).

Moreover, the zeros of $H(z)$ can be assigned by linear equations (linear constraints) of $a_1, \ldots, a_N$. Define $n_H(z) := z^N - \sum_{k=1}^{N} a_k z^{N-k}$. Then, $H(z)$ has M zeros at $z = z_0$ if and only if

$$\left.\frac{d^k n_H(z)}{dz^k}\right|_{z=z_0} = 0, \quad k = 0, 1, \ldots, M-1,$$

where $\frac{d^0 n_H(z)}{dz^0} := n_H(z)$. The LMI with these linear constraints can be also effectively solved.

2.3 Stability of quantized feedback systems

We here consider stability of $\Delta\Sigma$ modulators as non linear systems. Let a state space realization of Σ be $\{A_\Sigma, B, C\}$. Then the state space representation of the $\Delta\Sigma$ modulator shown in Fig. 1 is given by

$$\begin{aligned} \sigma x &= (A_\Sigma - BC)x + B(r - n), \\ y &= Cx + n, \\ n &= Q(Cx) - Cx. \end{aligned} \tag{2}$$

Since $R \in \mathcal{S}'$, all eigenvalues of $A := A_\Sigma - BC$ lie inside the unit circle. By the Lyapunov stability theory, there exist $X = X^T > 0$ and $Y = Y^T > 0$ such that

$$A^T X A - X = -Y. \tag{3}$$

We assume the following:

Assumption 1. *For the quantizer Q, there exist $M > 0$ and $\delta > 0$ such that if $|\psi| \le M$ then $|Q(\psi) - \psi| \le \delta$.*

Under this assumption, we have the following proposition.

Theorem 1. *Assume that*

$$\Omega\gamma \le \frac{M}{\|C\|} \tag{4}$$

where

$$\begin{aligned} \Omega &:= \|B\|(\|r\|_\infty + \delta), \\ \gamma &:= \max\{\alpha, \beta\}, \\ \alpha &:= \sqrt{\frac{|\overline{\lambda}(X) - \underline{\lambda}(Y)|\beta^2 + 2\|XA\|\beta + \overline{\lambda}(X)}{\underline{\lambda}(X)}}, \\ \beta &:= \frac{\|XA\| + \sqrt{\|XA\|^2 + \overline{\lambda}(X)\underline{\lambda}(Y)}}{\underline{\lambda}(Y)}. \end{aligned}$$

Then the region

$$\mathcal{R} := \left\{ x \in \mathbb{R}^n : x'Xx \le \underline{\lambda}(X)\frac{M^2}{\|C\|^2} \right\} \tag{5}$$

is an invariant set for the system (2).

Proof. Let x be in $\mathcal{R}$. Then, $x'Xx \le \underline{\lambda}(X)M^2/\|C\|^2$ and $|x| \le M/\|C\|$. It follows that $|Cx| \le M$ and by the assumption 1, we have $|n| = |Q(Cx) - Cx| \le \delta$. Put $w := B(r - n)$. Then we have

$$|w| \le \|B\|(\|r\|_\infty + \delta) = \Omega. \tag{6}$$

Define $V(x) := x'Xx$. Then

$$\begin{aligned}
&V(\sigma x) - V(x)\\
&\qquad = (Ax+w)'X(Ax+w) - x'Xx\\
&\qquad = -x'Yx + 2w'XAx + w'Xw\\
&\qquad \leq -\underline{\lambda}(Y)|x|^2 + 2\Omega\|XA\||x| + \overline{\lambda}(X)\Omega^2.
\end{aligned}$$

If $|x| \geq \Omega\beta$, we have $V(\sigma x) - V(x) \leq 0$, and hence $x \in \mathcal{R}$. If $|x| \leq \Omega\beta$,

$$\begin{aligned}
&(\sigma x)'X(\sigma x)\\
&\leq \left\{\overline{\lambda}(X) - \underline{\lambda}(Y)\right\}|x|^2 + 2\Omega\|XA\||x| + \overline{\lambda}(X)\Omega^2\right\}\\
&\leq \left\{|\overline{\lambda}(X) - \underline{\lambda}(Y)|\beta^2 + 2\|XA\|\beta + \overline{\lambda}(X)\right\}\Omega^2\\
&\leq \underline{\lambda}(X)\gamma^2\Omega^2\\
&\leq \underline{\lambda}(X)\frac{M^2}{\|C\|^2}.
\end{aligned}$$

It follows that $\sigma x \in \mathcal{R}$. $\square$

By this theorem, taking $x(0) \in \mathcal{R}$ results the state $x(k)$ is bounded for all $k > 0$. In practice, we can set the initial state $x(0) := 0$, and hence the feedback system is stable in the sense of Theorem 1.

Example 1. *Consider the conventional $\Delta\Sigma$ modulator shown in Fig. 1 with $\Sigma(z) = 1/(1-z)$ and the quantizer*

$$Q(\psi) = \mathrm{sgn}(\psi) = \begin{cases} 1, & \psi > 0, \\ -1, & \psi < 0. \end{cases} \tag{7}$$

Then we have $A_\Sigma = 1$, $B = 1$, $C = 1$, and $M = 2$, $\delta = 1$. Since $A = A_\Sigma - BC = 0$, the Lyapunov equation (3) is satisfied for arbitrary $X > 0$ and $Y > 0$. By Theorem 1, if $\|r\|_\infty \leq 1$ and $|x(0)| \leq 2$, then $|x(k)| \leq 2$ for all $k \geq 0$.

3 Design of $\Delta\Sigma$ AD/DA Converters

By using the controller characterized in Lemma 1, we here design AD or DA converters with $\Delta\Sigma$ modulators. To take account of intersample response, we introduce sampled-data H^∞ optimization. From the input/output relation (1), the system from n to y is linear in R, and the design problem is formulated as a one-block sampled-data H^∞ optimization. Then, via fast discretization method [3], we reduce the problem to a discrete-time H^∞ optimization.

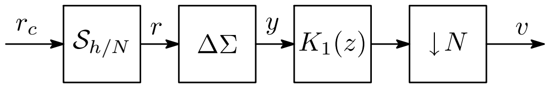


Figure 3: Oversampling $\Delta\Sigma$ AD converter

3.1 Design problem of $\Delta\Sigma$ AD converters

Fig. 3 shows an oversampling $\Delta\Sigma$ AD (analog-to-digital) converter. The input analog signal r_c is sampled by the sampler $\mathcal{S}_{h/N}$ with sampling time h/N, and becomes a discrete-time signal r. Then, a $\Delta\Sigma$ modulator modulate the signal r to become another digital signal y, whose word length is usually 1 [bit]. The signal y is downsampled by the *decimator* [9] $(\downarrow N)H(z)$ to become a discrete-time v with sampling time h. The filter $K_1(z)$ is called a *decimation filter*, which eliminates aliases that the *downsampler* $\downarrow N$ will cause. By the downsampler $\downarrow N$, the sampling time is reduced to h by taking every N-th output sample of the filter $K_1(z)$. Finally, we have a digital signal v with sampling time h and word length b [bit].

Our objective here is to design the decimation filter $K_1(z)$ to reconstruct the original analog signal r_c. Conventionally, the filter is designed assuming that the analog signal r_c is fully band-limited by the Nyquist frequency π/h. It follows that the filter $K_1(z)$ is designed to approximate the ideal low-pass filter whose cutoff frequency is π/h [9]. The real analog signals, however, have the spectra beyond the Nyquist frequency, and hence this design will not be necessarily optimal.

Therefore, we propose an alternative method that takes account of the analog performance, in particular, the frequency component over the Nyquist frequency. For this purpose, we consider the error system shown in Fig. 4. In the figure, we take the linear model for the $\Delta\Sigma$ modulator. The delay e^{-ds} in the

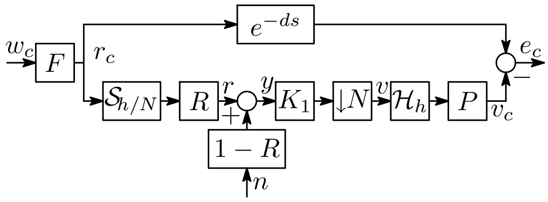


Figure 4: Error system $\mathcal{T}_1$ for AD converter

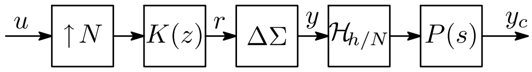


Figure 5: Oversampling $\Delta\Sigma$ DA converter

upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. To make the design easy, we take the linear model 1 for the $\Delta\Sigma$ modulator.

Let $\mathcal{T}_1 = [\mathcal{T}_{1w}, \mathcal{T}_{1n}]$ denote the input/output operators from the input $[w_c, n]^T$ to the error e_c. Our design problem is then as follows:

Problem 2. *Given a stable, strictly proper $F(s)$, stable, proper $P(s)$, stable, strictly proper $R(z)$, upsampling factor N, delay d, sampling period h, find $K_1(z)$ which minimizes*

$$\|\mathcal{T}_1\|_\infty := \sup_{w_c \in L^2, n \in \ell^2} \frac{\|e_c\|_{L^2}}{\sqrt{\|w_c\|_{L^2}^2 + \|n\|_{\ell^2}^2}}.$$

3.2 Design problem of $\Delta\Sigma$ DA converters

Fig. 5 shows an oversampling $\Delta\Sigma$ DA (digital-to-analog) converter. Assume that the input signal u has sampling time h and word length b [bits]. The digital signal u is first upsampled by a *interpolator* [9] $(\uparrow N)K_2(z)$. By $\uparrow N$, $N-1$ zeros are introduced between two consecutive input values [9]. The following digital filter $K_2(z)$, called *interpolation filter*, operates on the $N-1$ zero-valued samples inserted by $\uparrow N$ to yield nonzero values between the original samples.

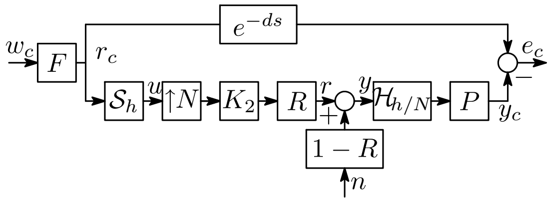


Figure 6: Error system $\mathcal{T}_2$ for DA converter

Then the interpolated signal r goes through a $\Delta\Sigma$ modulator, and becomes a signal y whose word length is converted to another one, by usual 1 [bit]. Then the discrete-time signal y is converted to a continuous signal by the zero order hold $\mathcal{H}_{h/N}$ with hold time h/N, smoothed by a continuous-time filter $P(s)$, and finally becomes an analog signal y_c.

Our objective here is to design the interpolation filter $K_2(z)$ to interpolate samples taking account of the analog performance. If we a priori have the knowledge about the characteristic of the original analog signal (e.g., u is a sampled data of an orchestral music), we can use it for reconstruction.

Therefore, in the same way as our design of AD converters, we consider the error system Fig. 6 for designing the filter $K_2(z)$. Obviously, $K_2(z)$ cannot control the quantization noise, and hence we here design not only $K_2(z)$ but also $R(z)$. Our design problem is then as follows:

Problem 3. *Given a stable, strictly proper $F(s)$, stable, proper $P(s)$, upsampling factor N, delay L, sampling period h, find $K_2(z)$ and $R(z)$ which minimizes*

$$\|\mathcal{T}_{2w}\|_\infty := \sup_{w_c \in L^2} \frac{\|e_c\|_{L^2}}{\|w_c\|_{L^2}}$$

and

$$\|\mathcal{T}_{2n}\|_\infty := \sup_{n \in \ell^2} \frac{\|e_c\|_{L^2}}{\|n\|_{\ell^2}}.$$

Note that $\mathcal{T}_{2n}$ depends only on $R(z)$ while $\mathcal{T}_{2w}$ on $K_2(z)$ and $R(z)$. To obtain $K_2(z)$ and $R(z)$, we first optimize $\|\mathcal{T}_{2n}\|_\infty$ for $R(z)$, then we fix $R(z)$ and optimize $\|\mathcal{T}_{2w}\|_\infty$ for $K_2(z)$.

3.3 Fast discretization of sampled-data systems

Problem 2 and 3 involve a continuous-time delay component e^{-ds}, and hence they are infinite-dimensional sampled-data problems. To avoid this difficulty, we employ the fast discretizaition method [3]. By this method, our design problems are approximated by finite-dimensional discrete-time problems assuming that the delay time d is mh, $m \in \mathbb{N}$.

First, we define the *discrete-time lifting* (or *blocking*) operator [1].

Definition 1. *Define the discrete-time lifting* $\mathbb{L}_N$ *and its inverse* $\mathbb{L}_N^{-1}$ *by*

$$\begin{aligned}\mathbb{L}_N &:= (\downarrow N)\begin{bmatrix}1 & z & \dots & z^{N-1}\end{bmatrix}^T,\\ \mathbb{L}_N^{-1} &:= \begin{bmatrix}1 & z^{-1} & \dots & z^{-N+1}\end{bmatrix}(\uparrow N).\end{aligned}$$

Then, the fast-discretized system T_1 of the sampled-data system $\mathcal{T}_1$ is given as follows.

$$\begin{aligned}T_1(z) &= \begin{bmatrix}T_{1w}(z) & T_{1n}(z)\end{bmatrix},\\ T_{1w}(z) &= \{z^{-m} - \widetilde{P_d}(z)H\widetilde{K_1}(z)\widetilde{R}(z)\}\widetilde{F_d}(z),\\ T_{1n}(z) &= -\widetilde{P_d}(z)H\widetilde{K_1}(z)\{I - \widetilde{R}(z)\},\end{aligned}$$

where $\widetilde{F_d}$, $\widetilde{P_d}$, $\widetilde{R}$ and $\widetilde{K_1}$ are discrete-time LTI systems given by

$$\begin{aligned}\widetilde{F_d} &:= \mathbb{L}_N F_d \mathbb{L}_N^{-1}, \quad F_d := \mathcal{S}_{T/N} F \mathcal{H}_{T/N}\\ \widetilde{P_d} &:= \mathbb{L}_N P_d \mathbb{L}_N^{-1}, \quad P_d := \mathcal{S}_{T/N} P \mathcal{H}_{T/N},\\ \widetilde{R} &:= \mathbb{L}_N R \mathbb{L}_N^{-1},\\ \widetilde{K_1} &:= \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}\mathbb{L}_N K_1 \mathbb{L}_N^{-1},\end{aligned}$$

and H is a matrix $H := \begin{bmatrix}1 & \dots & 1\end{bmatrix}^T$.

In the same way, the fast-discretized systems of $\mathcal{T}_{2w}$ and $\mathcal{T}_{2n}$ are given as follows.

$$\begin{aligned}T_{2w}(z) &= \{z^{-m} - \widetilde{P_d}(z)\widetilde{R}(z)\widetilde{K_2}(z)S\}\widetilde{F_d}(z),\\ T_{2n}(z) &= -P_d(z)\{1 - R(z)\},\end{aligned}$$

where $\widetilde{K_2}(z)$ is a discrete-time LTI systems given by

$$\widetilde{K_2} := \mathbb{L}_N K_2 \mathbb{L}_N^{-1}\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}^T,$$

and S is a matrix $S := \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$.

By using these formulae, our problems are reduced to discrete-time H^∞ optimization ones. By discrete-time H^∞ optimization, we can obtain the optimal $\widetilde{K_1}(z)$, $\widetilde{K_2}(z)$ and $R(z)$. We reconstruct $K_1(z)$ and $K_2(z)$ from $\widetilde{K_1}(z)$ and $\widetilde{K_2}(z)$ respectively, by the following equations.

$$\begin{aligned}K_1(z) &= z^{-N}\widetilde{K_1}(z^N)\begin{bmatrix}1 & z & \dots & z^{N-1}\end{bmatrix}^T,\\ K_2(z) &= \begin{bmatrix}1 & z^{-1} & \dots & z^{-N+1}\end{bmatrix}\widetilde{K_2}(z^N).\end{aligned}$$

4 Design Example

4.1 Design of AD converters

We here present a design example of $\Delta\Sigma$ AD converters. The design parameters are as follows: the sampling period $h = 1$, the upsampling ratio $N = 8$, the reconstruction delay $d = h/N = 1/8$ ($m = 1$), the filter in $\Delta\Sigma$ modulator $R(z) = z^{-1}$ (i.e., conventional $\Delta\Sigma$), and the analog filters are

$$\begin{aligned}F(s) &= \frac{1}{(Ts+1)(0.1Ts+1)}, \quad T = 22.05/\pi,\\ P(s) &= 1.\end{aligned}$$

Note that the lowpass filter $F(s)$ simulates the frequency energy distribution of a typical orchestral music, which are observed by FFT analysis of analog records of some orchestral music. For comparison, we compare our filter with the equiripple one obtained by Parks-McClellan method [9] of order 41. Parks-McClellan method is widely used for designing FIR filters.

Fig. 7 shows the frequency responses of the decimation filters. The equiripple filter shows the sharper decay beyond the cut-off frequency $\omega = \pi/2$, while the filter by sampled-data design shows a rather slow decay.

To see the difference, we simulate the oversampling $\Delta\Sigma$ AD converter shown in Fig. 3 with the one-bit quantizer given by (7) for Q. Fig. 8 shows the time responses against a rectangular wave. The sampled-data designed $\Delta\Sigma$ AD converter well reconstructs the rectangular wave, while the decimator

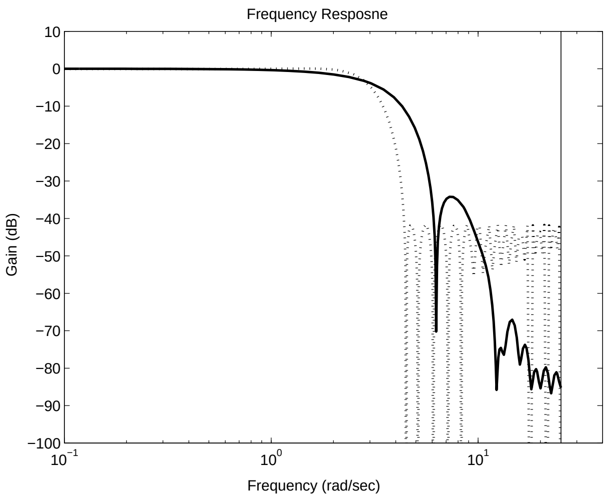


Figure 7: Decimation filters for AD converter: sampled-data H^∞ design (solid) and equiripple design (dotted)

with the equiripple filter exhibits an amount of ringing around the edges. This is because the equiripple design does not take account of higher frequency than the Nyquist frequency.

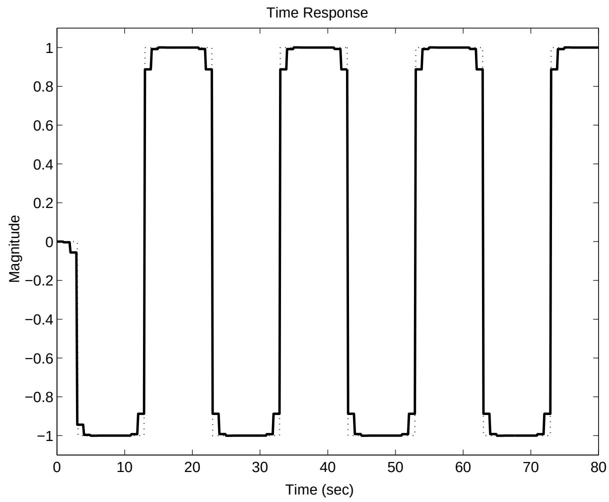
4.2 Design of DA converters

We here present a design example of $\Delta\Sigma$ DA converters. The design parameters are the same as those used above except for

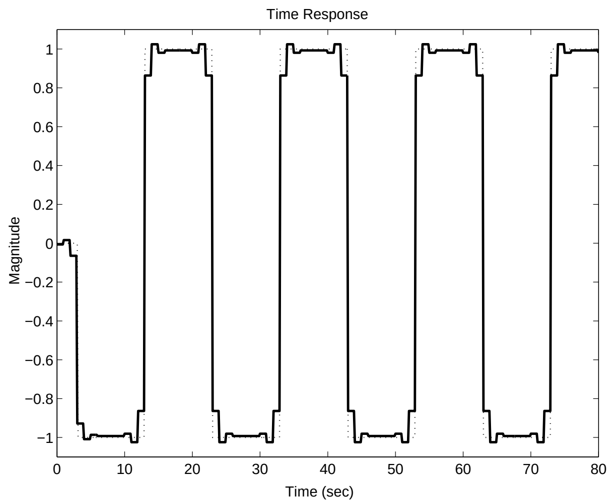
$$P(s) = \frac{1}{(T_c s + 1)^2}, \quad T_c := \frac{h}{\pi}.$$

We design the filter in $\Delta\Sigma$ modulator $R(z)$ and the interpolation filter $K_2(z)$ by the sampled-data H^∞ optimization. By assuming that $R(z)$ is an FIR filter of order 7, we design $R(z) \in \mathcal{S}'$ via LMI optimization [10]. For comparison, we take $R(z) = z^{-1}$ and the equiripple filter for $K_2(z)$ of order 21 as a conventional design.

The obtained interpolation filters are shown in Fig. 9. The gain around $\omega = 1/T_c = \pi$ of our filter is relatively large because the filter is designed by considering the lowpass characteristic of $P(s)$.



(a) Sampled-data H^∞ design



(b) Equiripple design

Figure 8: Time response

Then, we simulate the oversampling $\Delta\Sigma$ DA converter shown in Fig. 5 with the one-bit quantizer (7)

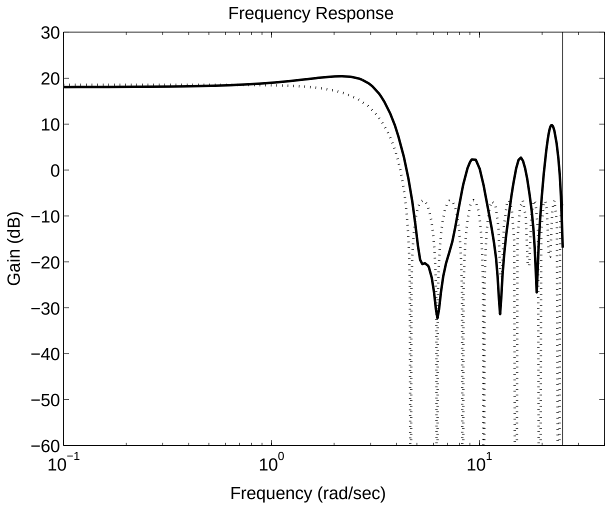


Figure 9: Interpolation filters for DA converter: sampled-data H^∞ design (solid) and equiripple design (dotted)

Table 1: Comparison of error

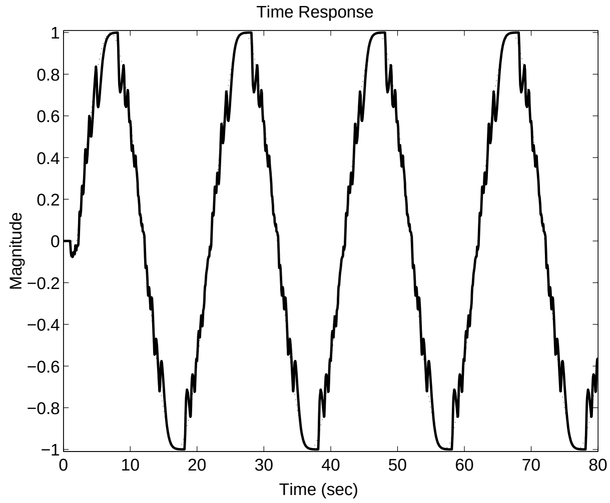
	Proposed	Conventional
$\|e_c\|_\infty$	2.08×10^{-1}	2.67×10^{-1}
$\|e_c\|_2$	5.68×10^{-1}	7.21×10^{-1}
$\mathrm{RMS}(e_c)$	6.34×10^{-2}	8.06×10^{-2}

for Q. We take for the digital input u a sinusoidal wave

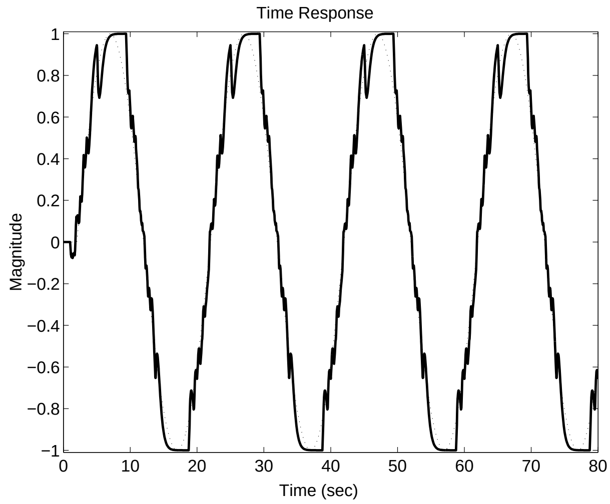
$$u[k] = \sin(0.1\pi k), \quad k = 0, 1, 2, \ldots, 80.$$

The time responses are shown in Fig. 10. The conventional DA converter shows large errors around $t = 10, 20, \ldots$. To see the difference, we show the absolute error in Fig. 11, and some norms of the error in Table 1. In the table, RMS is the root-mean-square values defined as follows (cf. the definition of pow): For fixed $T_f > 0$, RMS is defined as

$$\mathrm{RMS}(\phi_c) := \left\{ \frac{1}{T_f} \int_0^{T_f} |\phi_c(t)|^2 dt \right\}^{\frac{1}{2}}.$$



(a) Sampled-data H^∞ design



(b) Equiripple design

Figure 10: Time response

These comparisons show that our design is superior to the conventional one.

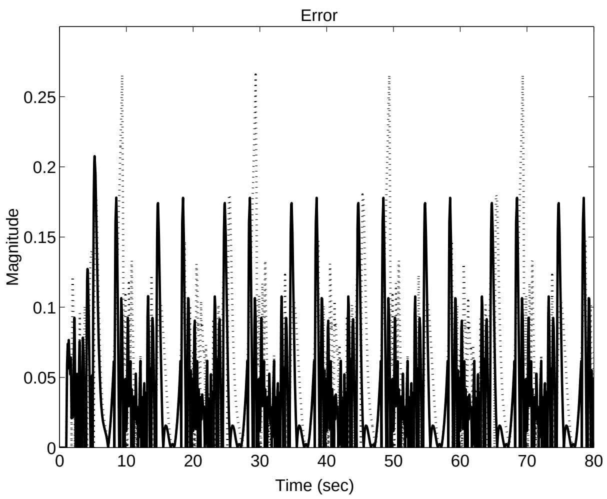


Figure 11: Error: sampled-data H^∞ design (solid) and equiripple design (dotted)

5 Conclusions

We have proposed a new design method for $\Delta\Sigma$ converters via sampled-data H^∞ optimization. We have also discussed the stability and the performance of $\Delta\Sigma$ converters as nonlinear systems. We have presented design examples and shown the advantages of the present method.

A further direction of this study will be to apply sampled-data control to the MSOC converters [8]. Their converter show a better performance than conventional ones, and sampled-data optimization will improve the performance further.

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