Multirate Signal Reconstruction and Filter Design via Sampled-Data H^∞ Control

Yutaka Yamamoto[1] Masaaki Nagahara[2] Hisaya Fujioka[3]

Abstract

This paper studies the problem of digital signal reconstruction in the multirate framework. In contrast to the typical digital domain formulation in the current digital signal processing, we present a solution that optimizes an H^∞ analog performance, via the modern sampled-data control. While the standard technique often indicates that an ideal digital low-pass filter is preferred, we show that the optimal solution need not be an ideal low-pass when the signal is not completely band-limited. The present method also suggests a new filter design method without recourse to analog filter designs. A design example is presented to show the advantages of the present method.

1 Introduction

Multirate techniques are now quite popular in the digital signal processing. They are particularly effective in subband coding, and various techniques for economical information saving has been developed [3, 9, 10].

They are also standard in signal decoding in audio/speech processing. For example, in the commercial CD format, the sampling frequency is 44.1 kHz, but one hardly employs the same sampling period in decoding. A popular technique is to first *upsample* the encoded digital signal, cut the parasitic imaging components via a digital low-pass filter, and then convert it back to an analog signal with a hold device and an analog low-pass filter. The chief advantage here is that one can employ a fast hold device, and need not use a very sharp analog filter (thereby avoiding much phase distortion induced by a sharp analog filter).

In the existing literature, it is a commonly accepted principle that one inserts a very sharp digital low-pass filter after the upsampler to eliminate the effect of imaging components [9, 10]. This is based on the following reasoning: Suppose that the original signal is fully band-limited. Then the imaging components induced by upsampling is not relevant to the original signal and hence they must be removed by a low-pass filter. If the original signal is band-limited, the closer is this filter to an ideal filter, the better.

In practice, however, no signals are entirely band-limited in a practical range of a passband, and they obey only an approximate frequency characteristic. The argument above is thus valid only in an approximate sense. One may rephrase this as a problem of robustness: namely, when the original signals are not fully band-limited but obey only a certain frequency characteristic, how close should the digital filter be to the ideal low-pass filter?

This type of question has been seldom addressed in the signal processing literature until very recently. However, this can be properly placed in the framework of sampled-data control, and there are now several investigations that apply the sampled-data control methodology to digital signal processing. Among them, Chen and Francis [2] solves the design of multirate filter banks in the discrete-time H^∞ setting; Khargonekar and Yamamoto [5] formulates and solves a single-rate signal reconstruction problem with optimal analog H^∞ performance. This has been generalized in [6, 7] to a multirate context. A multirate D/A conversion has been studied in [4]

We will formulate a digital signal reconstruction problem under the following assumptions:

- the orignal analog signal is subject to a certain frequency characteristic, but not fully band-limited;
- the digital signal can be upsampled to employ a faster hold device;
- Overall analog H^∞ performance must be optimized.

This may also be regarded as an optimal D/A converter design. We will show that performance improvement is possible over a conventional low-pass filter. It is also seen that presented method can be used a new design method for a low-pass filter.

[1]Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, JAPAN Email: yy@i.kyoto-u.ac.jp; all correspondence should be addressed to the first author.

[2]nagahara@acs.i.kyoto-u.ac.jp

[3]fujioka@i.kyoto-u.ac.jp

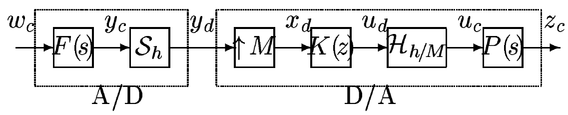

Figure 1: Multirate Signal Reconstruction

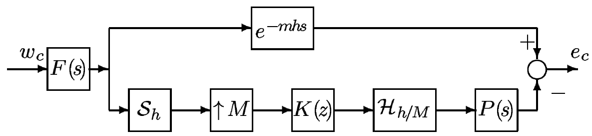

Figure 2: Signal reconstruction error system

2 Problem Formulation

Consider the block diagram Fig. 1. The incoming signal w_c first goes through an anti-aliasing filter $F(s)$ and the filtered signal y_c becomes nearly (but not entirely) band-limited. $F(s)$ governs the frequency-domain characteristic of the analog signal y_c. This signal is then sampled by $\mathcal{S}_h$ to become a discrete-time signal y_d with sampling period h. This signal is usually stored or transmitted with some media (e.g., CD) or a channel.

To restore y_c we usually let it pass through a digital filter, a hold device and then an analog filter. The present setup however places yet one more step: The discrete-time signal y_d is first upsampled by $\uparrow M$:

$$\uparrow M : y_d \mapsto x_d : x_d[k] = \begin{cases} y_d[l], & k = Ml,\ l = 0, 1, \ldots \\ 0, & \text{otherwise} \end{cases}$$

by factor M, and becomes another discrete-time signal x_d with sampling period h/M. The discrete-time signal x_d is then processed by a digital filter $K(z)$, becomes a continuous-time signal u_c by going through the 0-order hold $\mathcal{H}_{h/M}$ (that works in sampling period h/M), and then becomes the final signal by passing through an analog filter $P(s)$. An advantage here is that one can use a fast hold device $\mathcal{H}_{h/M}$ thereby making more precise signal restoration possible. The objective here is to design the digital filter $K(z)$ for given $F(s)$, M and $P(s)$.

Fig. 2 shows the block diagram for the error system for the design. The delay in the upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. Let T_{ew} denotes the input/output operator from w_c to $e_c := z_c(t) - u_c(t - mh)$. Our design objective is as follows:

Problem 1 *Given stable $F(s)$ and $P(s)$ and an attenuation level $\gamma > 0$, find a digital filter $K(z)$ such that*

$$\|T_{ew}\| := \sup_{w_c \in L^2[0,\infty)} \frac{\|T_{ew} w_c\|_2}{\|w_c\|_2} < \gamma. \tag{1}$$

3 Reduction to A Finite-Dimensional Problem

A difficulty in Problem 1 is that it involves a continuous time-delay, and hence it is an infinite-dimensional problem. Another difficulty is that it contains the upsampler $\uparrow M$, so that it makes the overall system time-varying.

Following the method of [5, 6], however, we can reduce this problem to a finite-dimensional single-rate problem:

Theorem 1 *There exist (finite-dimensional) discrete-time systems $G_1(z)$, $G_2(z)$ such that (1) is equivalent to*

$$\|z^{-m} G_1(z) - \widetilde{K}(z) G_2(z)\|_\infty < \gamma, \tag{2}$$

where $\widetilde{K}(z)$ is the discrete-time lifting of $K(z)$.

Proof: We first reduce the problem to a single-rate problem. Define the discrete-time lifting $\mathbf{L}_M$ and its inverse $\mathbf{L}_M^{-1}$ by

$$\begin{aligned} \mathbf{L}_M &:= (\downarrow M) \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{M-1} \end{bmatrix} \\ \mathbf{L}_M^{-1} &:= \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M+1} \end{bmatrix} (\uparrow M), \end{aligned}$$

where $\downarrow M$ denotes the downsampler

$$\downarrow M : x_d \mapsto y_d : y_d[k] = x_d[Mk].$$

Then $K(z)(\uparrow M)$ can be rewritten as

$$\begin{aligned} K(z)(\uparrow M) &= \mathbf{L}_M^{-1} \widetilde{K}(z) \\ \widetilde{K}(z) &:= \mathbf{L}_M K(z) \mathbf{L}_M^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \end{aligned}$$

$\widetilde{K}(z)$ is an LTI, single-input/M-output system that satisfies

$$K(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M+1} \end{bmatrix} \widetilde{K}(z^M).$$

Using the generalized hold $\widetilde{\mathcal{H}}_h$ defined by

$$\begin{aligned} \widetilde{\mathcal{H}}_h &: \quad l^2 \ni v \mapsto u \in L^2,\ \ u(kh + \theta) = \mathbf{H}(\theta) v[k] \\ &\qquad \theta \in [0, h),\ \ k = 0, 1, 2, \ldots \\ \mathbf{H}(\theta) &:= \begin{cases} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}, & \theta \in [0, h/M) \\ \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix}, & \theta \in [h/M, 2h/M) \\ \quad \ldots & \\ \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, & \theta \in [(M-1)h/M, h) \end{cases} \end{aligned}$$

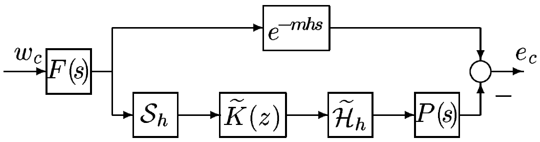

Figure 3: Reduced single-rate problem

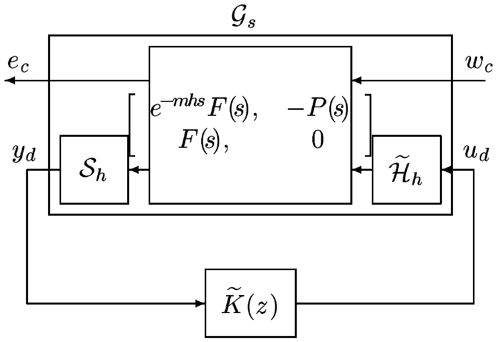

Figure 4: Sampled-data control system

we obtain the identity

$$\mathcal{H}_{h/M}\mathbf{L}_M^{-1} = \widetilde{\mathcal{H}}_h.$$

This yields

$$\mathcal{H}_{h/M}K(z)(\uparrow M)\mathcal{S}_h = \widetilde{\mathcal{H}}_h\widetilde{K}(z)\mathcal{S}_h.$$

Hence Fig. 2 is equivalent to Fig. 3. We can then invoke the technique of [5] to reduce this to a finite-dimensional design problem (2). ■

4 Approximation via Fast Sample/Hold

While the procedure above reduces Problem 1 to a finite-dimensional H^∞ problem, it is in general not numerically suitable for actual computation; the formulas are quite involved, and not so numerically tractable. It is often more convenient to resort to an approximation method. We employ the fast sample/hold approximation [1, 6]. This method approximates continuouos-time inputs and outputs via a sampler and hold that operate in the period h/M. The convergence of such an approximation is guaranteed in [8].

Note first that Fig. 3 yields the generalized plant formulation Fig. 4. We connect fast sample and hold devices $\mathcal{S}_{h/N}$, $\mathcal{H}_{h/N}$ with the plant as shown in Fig. 5. The resulting discrete-time approximant is given by the following formulas ($N = Ml$, l: positive integer):

$$G_{dN}(z) := \left[\begin{array}{c|cc} A_d & B_{N1} & B_{N2} \\ \hline C_{N1} & D_{N11} & D_{N12} \\ C_{c2} & 0 & 0 \end{array}\right]$$

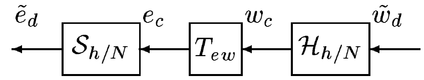

Figure 5: Fast discretization

$$\begin{aligned}
A_d &:= e^{A_c h}, \quad A_f := e^{A_c h/N} \\
& B_{2d} := \int_0^h e^{A_c t}B_{c2}dt \\
& [B_{1f}\ B_{2f}] := \int_0^{h/N} e^{A_c t}[B_{c1}\ B_{c2}]dt \\
B_{N1} &= \left[\ A_f^{N-1}B_{1f} \quad A_f^{N-2}B_{1f} \quad \cdots \quad B_{1f}\ \right] \\
B_{N2} &= \sum_{i=1}^{l}\left[\ A_f^{N-i}B_{2f} \quad A_f^{N-l-i}B_{2f} \quad \cdots \right. \\
& \qquad \left. \cdots \quad A_f^{N-(M-1)l-i}B_{2f}\ \right] \\
C_{N1} &= \begin{bmatrix} C_{c1} \\ C_{c1}A_f \\ \vdots \\ C_{c1}A_f^{N-1} \end{bmatrix} \\
D_{N11} &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_{c1}B_{1f} & 0 & \cdots & 0 \\ C_{c1}A_fB_{1f} & C_{c1}B_{1f} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ C_{c1}A_f^{N-2}B_{1f} & C_{c1}A_f^{N-3}B_{1f} & \cdots & 0 \end{bmatrix} \\
D_{N12} &= \begin{bmatrix} D_{c12} & 0 & \cdots & 0 \\ C_{c1}B_{2f} & D_{c12} & \cdots & 0 \\ C_{c1}A_fB_{2f} & C_{c1}B_{2f} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ C_{c1}A_f^{N-2}B_{2f} & C_{c1}A_f^{N-3}B_{2f} & \cdots & D_{c12} \end{bmatrix}\mathcal{I} \\
\mathcal{I} &:= \operatorname{diag}\{I_l\} \in \mathcal{R}^{N\times M}, \quad I_l := [1, 1, \ldots, 1]^T \in \mathcal{R}^l
\end{aligned}$$

$$\begin{bmatrix} F(s) & -P(s) \\ F(s) & 0 \end{bmatrix} =: \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & 0 & D_{c12} \\ C_{c2} & 0 & 0 \end{array}\right]$$

Then our design problem (1) is approximated as

$$\|z^{-m}G_{dN11}(z) + G_{dN12}(z)\widetilde{K}(z)G_{dN21}(z)\|_\infty < \gamma.$$

where

$$\begin{bmatrix} G_{dN11}(z) & G_{dN12}(z) \\ G_{dN21}(z) & 0 \end{bmatrix} = G_{dN}(z).$$

The resulting discrete-time problem is as depicted in Fig. 6.

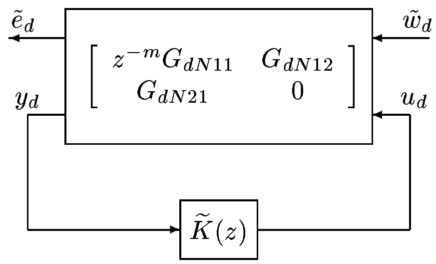

Figure 6: Discrete time system via FSFH

5 A Design Example

5.1 Design for Upsampling Factor $M = 4$

We first present a design example for

$$F(s) = \frac{1}{(10s+1)^2}, \quad P(s) = 1$$

with $h = 0.1$, $m = 2$ and upsampling factor $M = 4$. (In commercial CD players, M is usually $8 \sim 32$.) An approximate design is executed here for $N = M \times 4 = 16$.

Fig. 7 shows the (discrete-time) gain plots of three filters: $K_{SD}(z)$ designed by the present method, $K_{DT}(z)$ obtained by the simple discrete-time H^∞ design, and an FIR digital filter $K_L(z)$ by Lagrange interpolation.

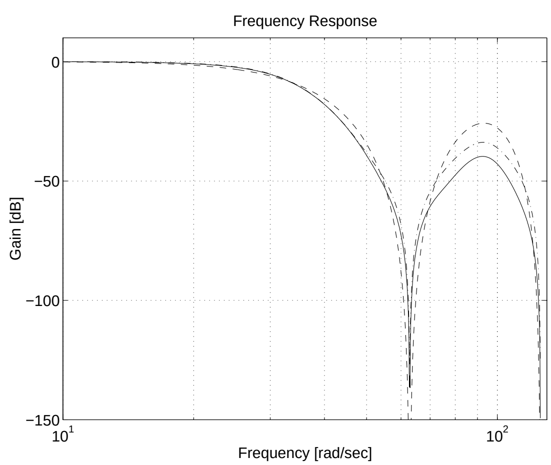

Figure 7: Frequency response of filter: $K_{SD}(z)$ (solid), $K_{DT}(z)$ (dash-dot) and $K_L(z)$ (dash)

The gain characteristics appear to be quite similar, although around 100 rad/sec, the present method shows more attenuation. The difference among them becomes clearer when we plot the gain plots of the respective error systems (Fig. 8)[1]. The present method exhibits a

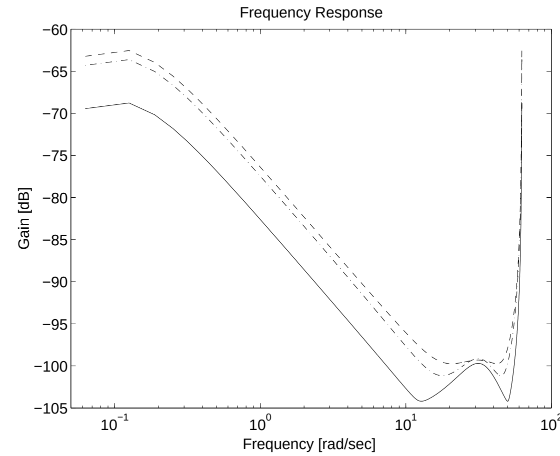

Figure 8: Frequency response of T_{ew} with $K_{SD}(z)$ (solid), $K_{DT}(z)$ (dash-dot) and $K_L(z)$ (dash)

Table 1: Orders of designed filters and $\|T_{ew}\|$

Filter	Order	$\|T_{ew}\|$
sampled data H^∞ IIR	15	3.8×10^{-4}
sampled data H^∞ FIR	19	3.8×10^{-4}
discrete time H^∞ IIR	15	6.9×10^{-4}
discrete time H^∞ FIR	19	6.9×10^{-4}
Lagrange filter	14	7.7×10^{-4}

clear advantage over all frequency range. Table 1 also shows the order and $\|T_{ew}\|$ of each filter.

Table 2 shows the (sub)optimal value of $\|T_{ew}\|$ for different M's. Larger M's result in better reconstruction results as naturally expected.

Fig. 9 and 10 show the time response against $w_c(t) = \sin 0.1t$ for the filter K_{SD} desinged for $M = 4$. They exhibit very high precision in reconstruction.

5.2 Design for Upsampling Factor $M = 2$

For comparison, we also present design results for $M = 2$ and compare it with the Johnston filter of order 31, which is often used in commercial applications.

As above, our sampled-data design has been executed

Table 2: Upsampling factor M and $\|T_{ew}\|$

M	$\|T_{ew}\|$
1	1.4×10^{-3}
2	7.4×10^{-4}
4	3.8×10^{-4}
6	2.5×10^{-4}
8	1.9×10^{-4}

[1] Note that the Nyquist frequency here corresponds to the original sampling period $h = 0.1$, and hence is 31.4 [rad/sec], whereas Fig. 7 the range is much wider corresponding to the upsampling of $M = 4$

for

$$F(s) = \frac{1}{(10s+1)^2}, \quad P(s) = 1$$

with $h = 0.1$, $m = 2$, with the difference: $M = 2$.

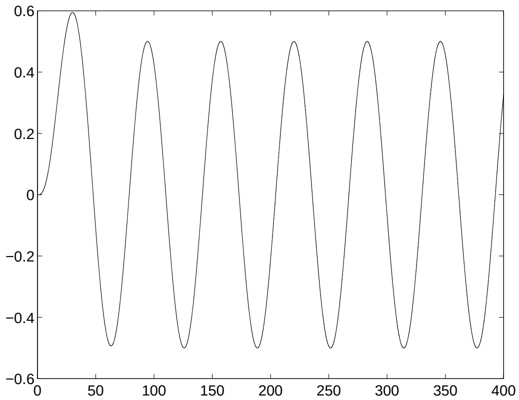

Figure 9: Time response of $z_c(t)$

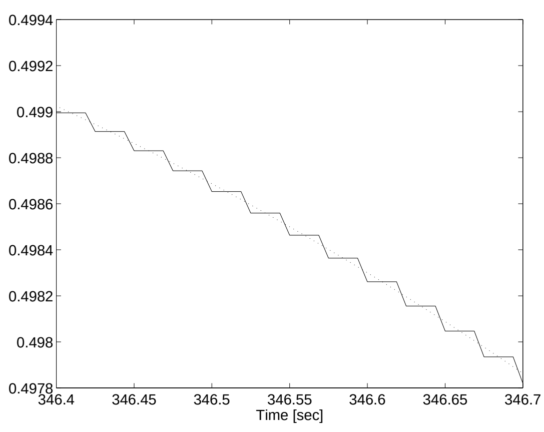

Figure 10: SD design: $z_c(t)$(solid) and $u_c(t - mh)$ (dot)

The obtained (sub)optimal filter $K_{SD}(z)$ is of order 7. Just for comparison, we have also obtained a purely discrete-time design $K_{DT}(z)$ which is again of order 7. $K_L(z)$ denotes the Lagrange filter of order 30, and $K_J(z)$ is the Johnston filter of order 31.

Fig. 11 shows the gain characteristics of these filters. The Johnston filter shows the sharpest decay beyond the cutoff frequency (π/h [rad/sec]) and the sampled-data design shows a rather slow decay. On the other

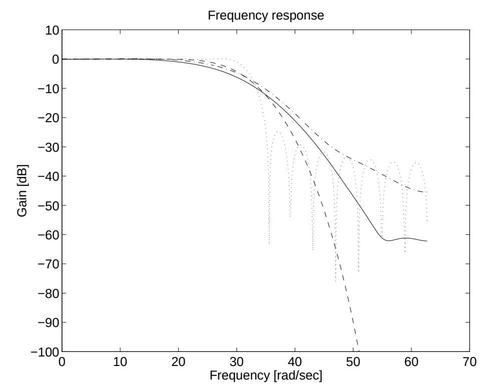

Figure 11: Frequency response of filters designed by sampled-data H^∞ synthesis K_{SD} (solid), discrete-time H^∞ synthesis K_{DT}(dash), Lagrange filter K_L(dash-dot) and Johnston filter K_J(dot)

hand, the reconstruction error characteristic in Fig. 12 exhibits quite an admirable performance in spite of the low-order of $K_{SD}(z)$ and small upsampling factor. It is almost comparable with 31st order Johnston filter.

While for those frequencies close to the cut-off the gain characteristic of the sampled-data design is not as good as the Johnston filter, the sampled-data designed filter need not be inferior. To see this, let us see the time responses against rectangular waves in Figs. 13, 14:

The Johnston filter exhibits a very typical Gibbs phenomenon, whreas the one by $K_{SD}(z)$ has much less peak around the edge. We also note that $K_{SD}(z)$ is nearly linear phase, as shown in Fig. 15.

6 Concluding Remarks

We have presented a new method of designing a digital filter in multirate signal reconstruction problem.

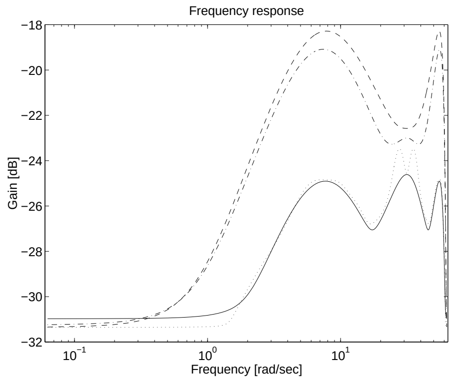

Figure 12: Frequency response of error system T_{ew} : sampled-data H^∞ synthesis (solid), discrete-time H^∞ synthesis (dash), Lagrange filter (dash-dot), Johnston filter (dot)

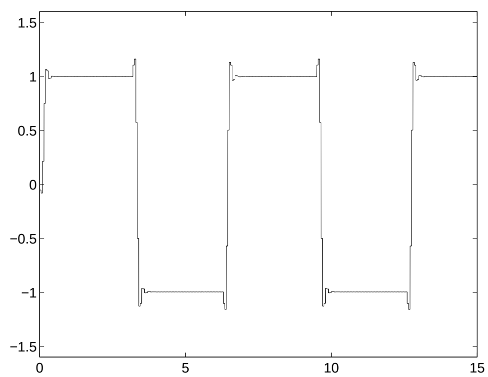

Figure 13: Time response (sampled-data syn.): $z_c(t)$ (solid), $u_c(t-mh)$ (dot)

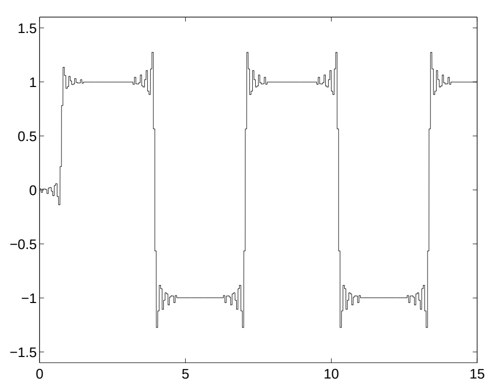

Figure 14: Time response (Johnston filter): $z_c(t)$ (solid), $u_c(t-mh)$ (dot)

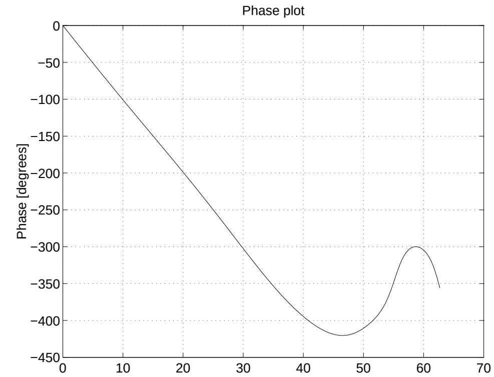

Figure 15: Phase plot of K_{SD}

About 6-8 dB improvement is accomplished in comparison with a typical digital filter.

References

[1] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*, Springer, New York (1995)

[2] T. Chen and B. A. Francis, "Design of multirate filter banks by $\mathcal{H}_\infty$ opimization," *IEEE Trans. Signal Processing*, **SP-43**: 2822–2830 (1995)

[3] N. J. Fliege, *Multirate Digital Signal Processing*, John Wiley, New York (1994)

[4] H. Ishii and Y. Yamamoto, "Sampled-data H^∞ and H^2/H^∞ design of multirate D/A converter," *Systems/Information/Control*, vol. 11, No. 10, pp. 586–593 (1998) (in Japanese)

[5] P. P. Khargonekar and Y. Yamamoto, "Delayed signal reconstruction using sampled-data control," *Proc. of 35th Conf. on Decision and Control*: 1259–1263, 1996.

[6] Y. Yamamoto, H. Fujioka and P. P. Khargonekar, Signal reconstruction via sampled-data control with multirate filter banks, *Proc. 36th Conf. on Decision and Control*, pp. 3395–3400 (1997)

[7] Y. Yamamoto and P. P. Khargonekar, "From sampled-data control to signal processing," in *Learning, Control and Hybrid Systems*, Springer Lecture Notes in Control and Information Sciences, vol. 241, pp. 108–126 (1998)

[8] Y. Yamamoto, A. G. Madievski and B. D. O. Anderson, "Approximation of frequency response for sampled-data control systems," *Automatica*, vol. 35, No. 4, pp. 729-734, 1999.

[9] P. P. Vidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, Englewood Cliffs, (1993)

[10] G. Zelniker and F. J. Taylor, *Advanced Digital Signal Processing: Theory and Applications*, Marcel Dekker (1994)