Multirate Signal Reconstruction and Filter Design via Sampled-Data $H^\infty$ Control

Yutaka Yamamoto\textsuperscript{1} Masaaki Nagahara\textsuperscript{2} Hisaya Fujioka\textsuperscript{3}

Abstract

This paper studies the problem of digital signal reconstruction in the multirate framework. In contrast to the typical digital domain formulation in the current digital signal processing, we present a solution that optimizes an $H^\infty$ analog performance, via the modern sampled-data control. While the standard technique often indicates that an ideal digital low-pass filter is preferred, we show that the optimal solution need not be an ideal low-pass when the signal is not completely band-limited. The present method also suggests a new filter design method without recourse to analog filter designs. A design example is presented to show the advantages of the present method.

1 Introduction

Multirate techniques are now quite popular in the digital signal processing. They are particularly effective in subband coding, and various techniques for economical information saving has been developed \cite{3, 9, 10}.

They are also standard in signal decoding in audio/speech processing. For example, in the commercial CD format, the sampling frequency is 44.1 kHz, but one hardly employs the same sampling period in decoding. A popular technique is to first upsample the encoded digital signal, cut the parasitic imaging components via a digital low-pass filter, and then convert it back to an analog signal with a hold device and an analog low-pass filter. The chief advantage here is that one can employ a fast hold device, and need not use a very sharp analog filter (thereby avoiding much phase distortion induced by a sharp analog filter).

In the existing literature, it is a commonly accepted principle that one inserts a very sharp digital low-pass filter after the upsampler to eliminate the effect of imaging components \cite{9, 10}. This is based on the following reasoning: Suppose that the original signal is fully band-limited. Then the imaging components induced by upsampling is not relevant to the original signal and hence must be removed by a low-pass filter. If the original signal is band-limited, the closer this filter is to an ideal filter, the better.

In practice, however, no signals are entirely band-limited in a practical range of a passband, and they obey only an approximate frequency characteristic. The argument above is thus valid only in an approximate sense. One may rephrase this as a problem of robustness: namely, when the original signals are not fully band-limited but obey only a certain frequency characteristic, how close should the digital filter be to the ideal low-pass filter?

This type of question has been seldom addressed in the signal processing literature until very recently. However, this can be properly placed in the framework of sampled-data control, and there are now several investigations that apply the sampled-data control methodology to digital signal processing. Among them, Chen and Francis \cite{2} solves the design of multirate filter banks in the discrete-time $H^\infty$ setting; Khargonekar and Yamamoto \cite{5} formulates and solves a single-rate signal reconstruction problem with optimal analog $H^\infty$ performance. This has been generalized in \cite{6, 7} to a multirate context. A multirate D/A conversion has been studied in \cite{4}

We will formulate a digital signal reconstruction problem under the following assumptions:

- the original analog signal is subject to a certain frequency characteristic, but not fully band-limited;
- the digital signal can be upsampled to employ a faster hold device;
- Overall analog $H^\infty$ performance must be optimized.

This may also be regarded as an optimal D/A converter design. We will show that performance improvement is possible over a conventional low-pass filter. It is also seen that presented method can be used a new design method for a low-pass filter.

\textsuperscript{1}Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, JAPAN Email: yys1.kyoto-u.ac.jp; all correspondence should be addressed to the first author.\textsuperscript{2}nagahara@acs.i.kyoto-u.ac.jp \textsuperscript{3}fujio@acs.i.kyoto-u.ac.jp
Consider the block diagram Fig. 1. The incoming signal $w_c$ first goes through an anti-aliasing filter $F(s)$ and the filtered signal $y_c$ becomes nearly (but not entirely) band-limited. $F(s)$ governs the frequency-domain characteristic of the analog signal $y_c$. This signal is then sampled by $S_h$ to become a discrete-time signal $y_d$ with sampling period $h$. This signal is usually stored or transmitted with some media (e.g., CD) or a channel.

To restore $y_c$ we usually let it pass through a digital filter, a hold device and then an analog filter. The present setup however places yet one more step: The discrete-time signal $y_d$ is first upsampled by $\uparrow M$:

$$\uparrow M : y_d \mapsto x_d : x_d[k] = \begin{cases} y_d[l], & k = MI, \ l = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases}$$

by factor $M$, and becomes another discrete-time signal $x_d$ with sampling period $h/M$. The discrete-time signal $x_d$ is then processed by a digital filter $K(z)$, becomes a continuous-time signal $u_c$ by going through the 0-th order hold $\mathcal{H}_{h/M}$ (that works in sampling period $h/M$), and then becomes the final signal by passing through an analog filter $P(s)$. An advantage here is that one can use a fast hold device $\mathcal{H}_{h/M}$ thereby making more precise signal restoration possible. The objective here is to design the digital filter $K(z)$ for given $F(s)$, $M$ and $P(s)$.

Fig. 2 shows the block diagram for the error system for the design. The delay in the upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. Let $T_{ew}$ denotes the input/output operator from $w_c$ to $e_c := z_c(t) - u_c(t - mh)$. Our design objective is as follows:

**Problem 1** Given stable $F(s)$ and $P(s)$ and an attenuation level $\gamma > 0$, find a digital filter $K(z)$ such that

$$||T_{ew}|| := \sup_{w_c \in l^2} \frac{||T_{ew}w_c||_2}{||w_c||_2} < \gamma. \quad (1)$$

**3 Reduction to A Finite-Dimensional Problem**

A difficulty in Problem 1 is that it involves a continuous time-delay, and hence it is an infinite-dimensional problem. Another difficulty is that it contains the up-sampler $\uparrow M$, so that it makes the overall system time-varying.

Following the method of [5, 6], however, we can reduce this problem to a finite-dimensional single-rate problem:

**Theorem 1** There exist (finite-dimensional) discrete-time systems $G_1(z)$, $G_2(z)$ such that (1) is equivalent to

$$||z^{-M}G_1(z) - \widetilde{K}(z)G_2(z)||_\infty < \gamma; \quad (2)$$

where $\widetilde{K}(z)$ is the discrete-time lifting of $K(z)$.

**Proof:** We first reduce the problem to a single-rate problem. Define the discrete-time lifting $L_M$ and its inverse $L_M^{-1}$ by

$$L_M := (\downarrow M) \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{M-1} \end{bmatrix} (\uparrow M),$$

where $\downarrow M$ denotes the downsampler

$$\downarrow M : x_d \mapsto y_d : y_d[k] = x_d[kM].$$

Then $K(z)(\uparrow M)$ can be rewritten as

$$K(z)(\uparrow M) = L_M^{-1} \widetilde{K}(z)$$

where

$$\widetilde{K}(z) := L_M K(z) L_M^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$\widetilde{K}(z)$ is an LTI, single-input/$M$-output system that satisfies

$$K(z) = [ 1 \ z^{-1} \ \ldots \ z^{-M+1} ] \widetilde{K}(z^M).$$

Using the generalized hold $\tilde{\mathcal{H}}_h$ defined by

$$\tilde{\mathcal{H}}_h : l^2 \ni v \mapsto u \in l^2, \ u(kh + \theta) = H(\theta)v[k] \quad \theta \in [0,h), \ k = 0,1,2,\ldots$$

$$H(\theta) := \begin{cases} [ 1 \ 0 \ 0 \ \ldots \ 0 ], & \theta \in [0,h/M) \\ [ 0 \ 1 \ 0 \ \ldots \ 0 ], & \theta \in [h/M,2h/M) \\ \ldots \end{cases}, \quad \theta \in [(M-1)h/M,h]$$

$$\tilde{\mathcal{H}}_h := \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix},$$
we obtain the identity
\[ \mathcal{H}_{h/M} \mathbf{L}_M^{-1} = \tilde{\mathcal{H}}_h. \]
This yields
\[ \mathcal{H}_{h/M} K(z)(\uparrow M) S_h = \tilde{\mathcal{H}}_h \tilde{K}(z) S_h. \]
Hence Fig. 2 is equivalent to Fig. 3. We can then invoke the technique of [5] to reduce this to a finite-dimensional design problem (2). 

4 Approximation via Fast Sample/Hold

While the procedure above reduces Problem 1 to a finite-dimensional \( H^\infty \) problem, it is in general not numerically suitable for actual computation; the formulas are quite involved, and not so numerically tractable. It is often more convenient to resort to an approximation method. We employ the fast sample/hold approximation [1, 6]. This method approximates continuous-time inputs and outputs via a sampler and hold that operate in the period \( h/M \). The convergence of such an approximation is guaranteed in [8].

Note first that Fig. 3 yields the generalized plant formulation Fig. 4. We connect fast sample and hold devices \( S_{h/N}, \mathcal{H}_{h/N} \) with the plant as shown in Fig. 5. The resulting discrete-time approximant is given by the following formulas (\( N = M I, I : \text{positive integer} \)):

\[
G_{dN}(z) := \begin{bmatrix}
A_d & B_{N_1} & B_{N_2} \\
C_{N_1} & D_{N_11} & D_{N_12} \\
C_{N_2} & 0 & 0
\end{bmatrix}
\]

Then our design problem (1) is approximated as
\[
\|z^{-\tau} G_{dN11}(z) + G_{dN12}(z) \tilde{K}(z) G_{dN21}(z) \|_\infty < \gamma.
\]
where
\[
\begin{bmatrix}
G_{dN11}(z) & G_{dN12}(z) \\
G_{dN21}(z) & 0
\end{bmatrix} = G_{dN}(z).
\]
The resulting discrete-time problem is as depicted in Fig. 6.
5 A Design Example

5.1 Design for Upsampling Factor $M = 4$
We first present a design example for

$$F(s) = \frac{1}{(10s + 1)^2}, \quad P(s) = 1$$

with $h = 0.1$, $m = 2$ and upsampling factor $M = 4$. (In commercial CD players, $M$ is usually $8 \sim 32$.) An approximate design is executed here for $N = M \times 4 = 16$.

Fig. 7 shows the (discrete-time) gain plots of three filters: $K_{SD}(z)$ designed by the present method, $K_{DT}(z)$ obtained by the simple discrete-time $H^\infty$ design, and an FIR digital filter $K_L(z)$ by Lagrange interpolation.

The gain characteristics appear to be quite similar, although around 100 rad/sec, the present method shows more attenuation. The difference among them becomes clearer when we plot the gain plots of the respective error systems (Fig. 8)\(^1\). The present method exhibits a clear advantage over all frequency range. Table 1 also shows the order and $\|T_{ew}\|$ of each filter.

Table 2 shows the (sub)optimal value of $\|T_{ew}\|$ for different $M$'s. Larger $M$'s result in better reconstruction results as naturally expected.

Fig. 9 and 10 show the time response against $w_c(t) = \sin 0.1t$ for the filter $K_{SD}$ designed for $M = 4$. They exhibit very high precision in reconstruction.

5.2 Design for Upsampling Factor $M = 2$
For comparison, we also present design results for $M = 2$ and compare it with the Johnston filter of order 31, which is often used in commercial applications.

As above, our sampled-data design has been executed

\(^1\)Note that the Nyquist frequency here corresponds to the original sampling period $h = 0.1$, and hence is 31.4 [rad/sec], whereas Fig. 7 the range is much wider corresponding to the upsampling of $M = 4$

\[\begin{array}{|c|c|c|}
\hline
\text{Filter} & \text{Order} & \|T_{ew}\| \\
\hline
\text{sampled data } H^\infty \text{ IIR} & 15 & 3.8 \times 10^{-4} \\
\text{sampled data } H^\infty \text{ FIR} & 19 & 3.8 \times 10^{-4} \\
\text{discrete time } H^\infty \text{ IIR} & 15 & 6.9 \times 10^{-4} \\
\text{discrete time } H^\infty \text{ FIR} & 19 & 6.9 \times 10^{-4} \\
\text{Lagrange filter} & 14 & 7.7 \times 10^{-4} \\
\hline
\end{array}\]

\[\begin{array}{|c|c|}
\hline
\text{Table 2: Upsampling factor } M \text{ and } \|T_{ew}\| \\
\hline
M & \|T_{ew}\| \\
\hline
1 & 1.4 \times 10^{-3} \\
2 & 7.4 \times 10^{-4} \\
4 & 3.8 \times 10^{-4} \\
6 & 2.5 \times 10^{-4} \\
8 & 1.9 \times 10^{-4} \\
\hline
\end{array}\]
for

\[ F(s) = \frac{1}{(10s + 1)^2}, \quad P(s) = 1 \]

with \( h = 0.1, \ m = 2 \), with the difference: \( M = 2 \).

The obtained (sub)optimal filter \( K_{SD}(z) \) is of order 7. Just for comparison, we have also obtained a purely discrete-time design \( K_{DT}(z) \) which is again of order 7. \( K_L(z) \) denotes the Lagrange filter of order 30, and \( K_J(z) \) is the Johnston filter of order 31.

Fig. 11 shows the gain characteristics of these filters. The Johnston filter shows the sharpest decay beyond the cutoff frequency \((\pi/h \ [\text{rad/sec}])\) and the sampled-data design shows a rather slow decay. On the other hand, the reconstruction error characteristic in Fig. 12 exhibits quite an admirable performance in spite of the low-order of \( K_{SD}(z) \) and small upsampling factor. It is almost comparable with 31st order Johnston filter.

While for those frequencies close to the cut-off the gain characteristic of the sampled-data design is not as good as the Johnston filter, the sampled-data designed filter need not be inferior. To see this, let us see the time responses against rectangular waves in Figs. 13, 14:

The Johnston filter exhibits a very typical Gibbs phenomenon, whereas the one by \( K_{SD}(z) \) has much less peak around the edge. We also note that \( K_{SD}(z) \) is nearly linear phase, as shown in Fig. 15.

6 Concluding Remarks

We have presented a new method of designing a digital filter in multirate signal reconstruction problem.
**Figure 12:** Frequency response of error system $T_{ew}$: sampled-data $H^\infty$ synthesis (solid), discrete-time $H^\infty$ synthesis (dash), Lagrange filter (dash-dot), Johnston filter (dot)

**Figure 13:** Time response (sampled-data syn.): $z_c(t)$ (solid), $u_c(t - mh)$ (dot)

**Figure 14:** Time response (Johnston filter): $z_c(t)$ (solid), $u_c(t - mh)$ (dot)

**Figure 15:** Phase plot of $K_{SD}

About 6-8 dB improvement is accomplished in comparison with a typical digital filter.

**References**


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