SPARSELY-PACKETIZED PREDICTIVE CONTROL BY ORTHOGONAL MATCHING PURSUIT *

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Abstract. We study packetized predictive control, known to be robust against packet dropouts in networked systems. To obtain sparse packets for rate-limited networks, we design control packets via an ℓ^0 optimization, which can be effectively solved by orthogonal matching pursuit. Our formulation ensures asymptotic stability of the control loop in the presence of bounded packet dropouts.

1. Packetized Predictive Control. Let us consider the following discrete-time, LTI plant model with a scalar input:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k), \quad k \in \mathbb{N}_0, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1.1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}$ for $k \in \mathbb{N}_0$. We assume that (A, B) is reachable. We are interested in a networked control architecture where the controller communicates with the plant actuator through an erasure channel, see Fig. 1.1. This channel introduces bounded packet-dropouts. In packetized predictive control (PPC), as described, e.g., in [4, 2] at each time instant k, the controller uses the state $\boldsymbol{x}(k)$ of the plant (1.1) to calculate and send a control packet of the form

$$\boldsymbol{u}\big(\boldsymbol{x}(k)\big) \triangleq \big[u_0\big(\boldsymbol{x}(k)\big), u_1\big(\boldsymbol{x}(k)\big), \ldots, u_{N-1}\big(\boldsymbol{x}(k)\big)\big]^\top \in \mathbb{R}^N \tag{1.2}$$

to the plant input node.

To achieve robustness against packet dropouts, buffering is used. To be more specific, suppose that at time instant k the data packet $\boldsymbol{u}(\boldsymbol{x}(k))$ defined in (1.2) is successfully received at the plant input side. Then, this packet is stored in a buffer, overwriting its previous contents. If the next packet, $\boldsymbol{u}(\boldsymbol{x}(k+1))$, is dropped, then the plant input $u(k+1)$ is set to $u_1(\boldsymbol{x}(k))$, the second element of $\boldsymbol{u}(\boldsymbol{x}(k))$. The subsequent elements of $\boldsymbol{u}(\boldsymbol{x}(k))$ are then successively used until some control packet $\boldsymbol{u}(\boldsymbol{x}(k+\ell))$, $\ell \geq 2$, is successfully received, i.e., no dropout occurs.

2. Sparse Control Packet Design for Asymptotic Stability. In the PPC method discussed above, control packets $\boldsymbol{u}(\boldsymbol{x}(k))$ are transmitted at each time $k \in \mathbb{N}_0$ through an erasure channel. It is often the case that the bandwidth of the channel is limited, and hence one has to compress control packets to a smaller data size. To achieve an efficient compression of packets, we adopt a technique developed in *compressed sensing* [1] that considers the *sparsity* of a signal (or a vector). A sparse vector contains many 0-valued elements and can be highly compressed by only coding

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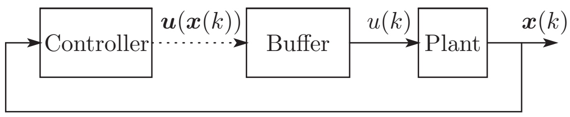


FIG. 1.1. *Networked Control System with PPC*

its few nonzero components. Based on this notion, in [2] we presented a sparsity-inducing ℓ^1/ℓ^2 optimization for PPC which gives *practical* stability, i.e., existence of a bounded invariant set. In the present work, we embellish the approach of [2] by showing how to find *sparse* control packets $\boldsymbol{u}(\boldsymbol{x}(k))$ ensuring that the networked system with bounded packet dropouts is *asymptotically stable*. For this, we propose to design the control packets via the following sparsity-promoting optimization:

$$\boldsymbol{u}(\boldsymbol{x}(k)) \triangleq \underset{\boldsymbol{u}\in\mathbb{R}^N}{\arg\min} \|\boldsymbol{u}\|_0 \text{ subject to } \|\boldsymbol{x}_{N|k}\|_P^2 + \sum_{i=0}^{N-1} \|\boldsymbol{x}_{i|k}\|_Q^2 \leq \boldsymbol{x}(k)^\top W \boldsymbol{x}(k), \quad (2.1)$$

where $\|\boldsymbol{u}\|_0$ is the number of the nonzero elements in $\boldsymbol{u} = [u_0, u_1, \dots, u_{N-1}]^\top$ and

$$\boldsymbol{x}_{0|k} = \boldsymbol{x},\ \boldsymbol{x}_{i+1|k} = A\boldsymbol{x}_{i|k} + Bu_i,\ i = 0, 1, \dots, N-1. \quad (2.2)$$

In (2.2), N is the horizon length, taken here equal to the buffer size. The matrix $Q > 0$ is a design parameter which allows one to shape the response of the plant state components; $P > 0$ and $W > 0$ are chosen such that the feedback system is asymptotically stable, as indicated in Theorem 2.1 below. To state our subsequent results, we introduce the matrices Φ, Υ and $\bar{Q}$ by

$$\Phi \triangleq \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix},\ \Upsilon \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},\ \bar{Q} \triangleq \text{blockdiag}\{\underbrace{Q, \dots, Q}_{N-1}, P\}.$$

THEOREM 2.1 (Asymptotic Stability). *Assume that the number of consecutive packet-dropouts is uniformly bounded by N. Suppose the matrices P, Q, and W are chosen by the following procedure:*

i. Choose $Q > 0$ arbitrarily.

ii. Solve the following Riccati equation to obtain $P > 0$:

$$P = A^\top PA - A^\top PB(B^\top PB)^{-1}B^\top PA + Q. \quad (2.3)$$

iii. Compute the constants $\rho \in [0, 1)$ and $c > 0$ via

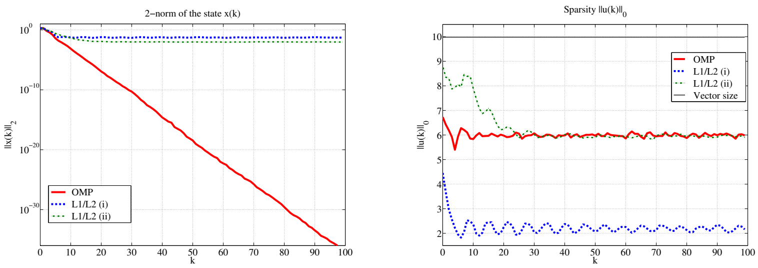
$$\rho \triangleq 1 - \lambda_{\min}(QP^{-1}),\ c \triangleq \frac{1-\rho^N}{1-\rho} \max_i \lambda_{\max}\{\Phi_i^\top P \Phi_i (\Phi^\top \bar{Q} \Phi)^{-1}\},$$

where Φ_i is the i-th column of matrix Φ.

iv. Choose a matrix $\mathcal{E}$ such that $0 < \mathcal{E} < (1-\rho)P/c$.

v. Set $W := P - Q + \mathcal{E}$.

Then for any choice of the control vector from the feasible set of the optimization (2.1) *(which includes the sparsest control packets $\boldsymbol{u}(\boldsymbol{x}(k))$), the networked control system is asymptotically stable, that is,* $\lim_{k\to\infty} \boldsymbol{x}(k) = \mathbf{0}$.



(a) Performance $\|\boldsymbol{x}(k)\|_2$ (b) Sparsity $\|\boldsymbol{u}(\boldsymbol{x}(k))\|_0$

FIG. 4.1. *Results by OMP (solid) and ℓ^1/ℓ^2 optimization (dash)*

3. Orthogonal Matching Pursuit. The optimization (2.1) can be rewritten as follows:

$$\boldsymbol{u}\big(\boldsymbol{x}(k)\big) \triangleq \mathop{\arg\min}_{\boldsymbol{u}\in\mathbb{R}^N} \|\boldsymbol{u}\|_0 \ \text{ subject to } \ \|G\boldsymbol{u} - H\boldsymbol{x}(k)\|_2^2 \le \boldsymbol{x}(k)^\top W \boldsymbol{x}(k), \tag{3.1}$$

where $G \triangleq \bar{Q}^{1/2}\Phi$ and $H \triangleq -\bar{Q}^{1/2}\Upsilon$. To solve this combinatorial optimization, we adopt an iterative greedy algorithm called *Orthogonal Matching Pursuit* (OMP) [3]. The algorithm is very simple and significantly faster than exhaustive search. Moreover, OMP always gives a vector in the feasible set of the optimization (2.1), and hence the networked control system will be asymptotically stable by Theorem 2.1.

4. Example. To show the effectiveness of the proposed method, we run 500 simulations with a fixed 4-th order unstable plant (the poles are -1.4396, $1.0808 \pm 0.6664i$, and 0.0220). The packet size N is 10. We generate a packet dropout at each time k with dropout probability $1/2$ (if there have been $N-1$ consecutive dropouts, we set the next dropout probability to be 0, i.e., the dropout process is Markovian). Figure 4.1 compares averaged results of the proposed method with that of [2]. Two plots (i) and (ii) are displayed for the method of [2], with different design parameters for sparsity. We can see that the OMP control exhibits asymptotic (even exponential) stability, whereas the ℓ^1/ℓ^2 method of [2] is only practically stable. The ℓ^1/ℓ^2 method produces much sparser vectors as in (i) than the OMP formulation, but the system does is not asymptotically stable even if one relaxes the sparsity constraint in the ℓ^1/ℓ^2 optimization as in (ii). In fact, it is proved in [2] that if the controlled plant is unstable, asymptotic stability is impossible by the ℓ^1/ℓ^2 method. The reason is that the ℓ^1/ℓ^2 method amounts to considering a fixed upper bound for the inequality constraint in (2.1) or (3.1) instead of time-varying $\boldsymbol{x}(k)^\top W\boldsymbol{x}(k)$. This leads to $\mathbf{0}$-valued control vector when $\|\boldsymbol{x}(k)\|_2$ is sufficiently small (note that $\boldsymbol{u} = \mathbf{0}$ is the sparsest vector among all vectors).

REFERENCES

[1] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, **52-4**, pp. 1289–1306, Apr. 2006.
[2] M. Nagahara and D. E. Quevedo, "Sparse representations for packetized predictive networked control," in *IFAC 18th World Congress*, Aug. 2011, pp. 84–89.
[3] Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad, "Orthogonal matching pursuit," in *Proc. the 27th Annual Asilomar Conf. on Signals, Systems and Computers*, Nov. 1993, pp. 40–44.
[4] D. E. Quevedo and D. Nešić, "Input-to-state stability of packetized predictive control over unreliable networks affected by packet-dropouts," *IEEE Trans. Autom. Control*, **56-2**, pp. 370–375, Feb. 2011.