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REPETITIVE CONTROL VIA SAMPLED-DATA H^{∞} CONTROL

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Abstract: Repetitive control has been implemented in various industrial applications to allow systems to track or reject unknown periodic signals with a fixed period. Repetitive controllers are usually implemented by digital computers, in which undesired ripples may occur between sampling instants. In this work, we approach this problem via sampled-data H^{∞} control. We show that our approach significantly attenuates the ripples without sacrificing the tracking performance.

Keywords: Repetitive control, Sampled-data control, Ripple attenuation

1. INTRODUCTION

Repetitive control (Nakano *et al.*, 1989) is a control to be designed for tracking periodic reference signals and for rejecting periodic disturbances. It has been widely implemented in various industrial applications. For example, high accuracy tracking in magnet power supply for proton synchrotron (Inoue *et al.*, 1981), control of robot manipulators which carry out repetitive tasks (Omata *et al.*, 1987), and so forth.

From a practical point of view, it is easier to implement the repetitive controllers digitally. Digital repetitive controllers can make the steady-state tracking error vanish at the sampling instants, due to the internal model principle (Nakano *et al.*, 1989). However, the output of the plant is still continuous-time, and it is well-known that digital repetitive control may result in undesired ripples between the sampling instants (Nakano *et al.*, 1989; Hara *et al.*, 1990).

There are several literatures on digital repetitive control systems investigating the ripples phenomena (Franklin and Emami-Naeini, 1986; Urikura and Nagata, 1987; Hara *et al.*, 1990). These researches are, however, in the discrete-time. To tackle the ripple phenomenon, we have to consider the intersample behavior. Therefore, we strongly propose to introduce *sampled-data control* (Chen and Francis, 1995) into repetitive control. Sampled-data control can take the intersample behavior into account, and hence the ripples in repetitive control systems can be naturally considered. From this point of view, several articles have studied repetitive control via sampled-data control (Langari and Francis, 1998; Ishii and Yamamoto, 1998). Langari and Francis propose to design the controller by using the induced power-norm (Langari and Francis, 1998), while Ishii and Yamamoto take the L^2 induced norm (i.e., H^{∞} norm) and the optimal filter is periodically time-varying (Ishii and Yamamoto, 1998). Based on these researches, we propose a design

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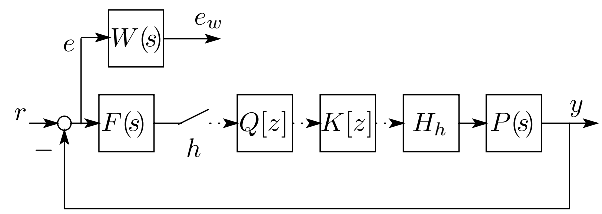

Fig. 1. Sampled data repetitive control system

method to find the H^∞ optimal linear time-invariant controller via sampled-data H^∞ control. Although sampled-data systems are generally infinite-dimensional, we introduce the fast-sampling/fast-hold approximation (Yamamoto *et al.*, 1999; Yamamoto *et al.*, 2002) to make the system finite-dimensional. By a numerical example, we show that our repetitive controller attenuates the intersample ripples, and simultaneously maintains excellent tracking performance.

2. PROBLEM FORMULATION

Consider the repetitive control system shown in Figure 1, where the signal r is the reference input and y is the plant output. In this block diagram, $P(s)$ is a continuous-time plant to be controlled, $F(s)$ an anti-aliasing filter, and $W(s)$ a frequency weight of the error e to be estimated. We assume that the plant $P(s)$ is strictly proper, the filter $F(s)$ and the weighting function $W(s)$ are strictly proper and stable. We have two controllers in the diagram; $Q[z]$ is the repetitive controller:

$$Q[z] = \frac{1}{z^M - 1} \tag{1}$$

and $K[z]$ is a digital controller (assumed to be proper and linear time-invariant) to stabilize the feedback system and to improve the performance. While the controllers are discrete-time systems, the plant is a continuous-time one, and hence we have the sampler S_h and the zero-order hold H_h where h is the sampling time.

Our objective here is to

- make the tracking error $e(t)$ vanish at the sampling instants $t = kh$ ($k \in \mathbb{Z}$) as $k \to \infty$,
- attenuate the intersample ripples between the sampling instants.

The former is achievable by the repetitive controller (1) because of the internal model principle (Nakano *et al.*, 1989). Define the filtered and sampled error $e_d[k] := (S_hFe)[k] = (Fe)(kh)$, $k = 0, 1, 2, \ldots$, then we have the following theorem:

Theorem 1. Assume that the controller $K[z]$ internally stabilizes the feedback system shown in Figure 1, and $M = T/h$ is a positive integer. Then, we have

$$\lim_{\substack{k\in\mathbb{Z}\\ k\to\infty}} e_d[k] = 0.$$

Remark 2. The sampled signal of $e(t)$, that is, $e(kh)$, $k = 0, 1, 2, \ldots$, will not vanish as $k \to \infty$. Let the reference signal $r(t)$ be a sinusoidal function $e^{j\omega_0 t}$, $\omega_0 := 2\pi/T$, then the steady state response of $e(t)$ is given as follows (Yamamoto and Araki, 1994):

$$\begin{aligned} e(t) = e^{j\omega_0 t} &- \frac{1}{h}\sum_{n=-\infty}^{\infty} P(j\omega_n)H_h(j\omega_n) \\ &\times S_d[e^{j\omega_0 h}]\widetilde{K}[j\omega_0 h]F(j\omega_0)e^{j\omega_n t}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} \widetilde{K}[z] &:= K[z]Q[z], \\ S_d[z] &:= \left\{1 + \widetilde{K}[z]G_d[z]\right\}^{-1}, \\ G_d[z] &:= (S_hPFH_h)[z], \\ H_h(s) &:= \frac{1-e^{-sh}}{s}, \\ \omega_n &:= \omega_0 + \frac{2\pi n}{h}, \quad n = 0, \pm 1, \pm 2, \ldots . \end{aligned}$$

Substituting $t = kh$, $k = 0, 1, 2, \ldots$ into the equation (2), we obtain

$$\begin{aligned} e(kh) &= e^{j\omega kh} \\ &\quad - \frac{F(j\omega_0)\sum_{n=-\infty}^{\infty} P(j\omega_n)H_h(j\omega_n)}{\sum_{n=-\infty}^{\infty} F(j\omega_n)P(j\omega_n)H_h(j\omega_n)} e^{j\omega kh}. \end{aligned}$$

If

$$\frac{F(j\omega_0)\sum_{n=-\infty}^{\infty} P(j\omega_n)H_h(j\omega_n)}{\sum_{n=-\infty}^{\infty} F(j\omega_n)P(j\omega_n)H_h(j\omega_n)} = 1, \tag{3}$$

then we have $e(kh) = 0$. For example, we have $e(kh) = 0$ if $F(s) = 1$, that is, we do not use any anti-aliasing filter, or if the plant is fully band-limited up to the Nyquist frequency $\omega_N := \pi/h$, that is,

$$P(j\omega) = 0, \quad |\omega| \geq \omega_N,$$

and the frequency of the input signal is $\omega_0 < \omega_N$. However, in general, the assumption (3) will hold, and hence, $e(kh)$ will not vanish as $k \to \infty$.

In order to attain the latter objective, we design the controller $K[z]$ via sampled-data control. Our design problem is as follows:

Problem 3. Let $\mathcal{T}$ be the system from r to e_w in Figure 1. Find the controller $K[z]$ (proper and linear time-invariant) which stabilizes the feedback system and minimizes the L^2 induced-norm of $\mathcal{T}$:

$$\|\mathcal{T}\| := \sup_{w\in L^2} \frac{\|\mathcal{T}w\|}{\|w\|}.$$

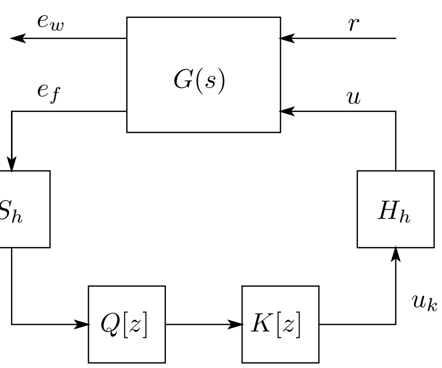

Fig. 2. Sampled data system $\mathcal{T}$

This problem is a sampled-data H^∞ optimal control problem. In the next section, we discuss how we should solve this problem.

3. SAMPLED-DATA H^∞ CONTROL FOR RIPPLE ATTENUATION

Define the state-space realizations of $P(s)$, $F(s)$ and $W(s)$ as

$$P(s) := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad F(s) := \left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & 0 \end{array}\right],$$
$$W(s) := \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & 0 \end{array}\right].$$

The feedback system in Figure 1 can be arranged as a generalized form as in Figure 2 with

$$G(s) = \begin{bmatrix} W(s) & -W(s)P(s) \\ F(s) & -F(s)P(s) \end{bmatrix} =: \left[\begin{array}{c|cc} A_c & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right]$$

where

$$A_c = \begin{bmatrix} A & 0 & 0 \\ -B_F C & A_F & 0 \\ -B_W C & 0 & A_W \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ B_F \\ B_W \end{bmatrix}, \quad B_2 = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 0 & C_W \\ C & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & C_F & 0 \end{bmatrix}.$$

Assume that (A_c, B_2) is stabilizable and (C_2, A_c) is detectable. Our objective here is to minimize the H^∞ norm of $\mathcal{T}$ (the system from r to e_w) in order to attenuate the intersample ripples.

To solve our sampled-data H^∞ problem, we employ the fast-sampling/fast-hold method (Yamamoto *et al.*, 1999; Yamamoto *et al.*, 2002). The fast-sampling/fast-hold technique is a method for approximating the performance of sampled-data systems. The procedure is as follows:

- discretize the continuous-time input by a hold with sampling period h/N,

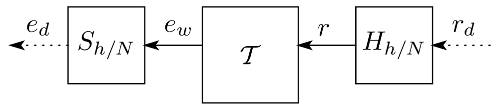

Fig. 3. Fast-sampling/fast-hold approximation

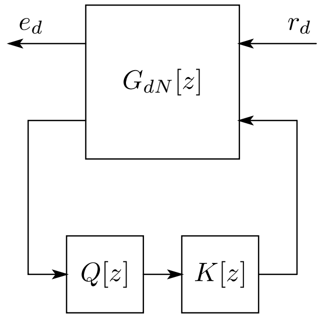

Fig. 4. Discretized system T_N

- discretize the continuous-time output by a sampler with sampling period h/N,

where N is a positive integer (see Figure 3). With large N, the discretized signals may be a good approximation of the continuous signals, and hence we can control intersample ripples. Moreover, we have the following theorem.

Theorem 4. For the sampled-data system $\mathcal{T}$ in Figure 2, there exist discrete-time systems $\{T_N : N = 1, 2, \dots\}$ such that

$$\lim_{N\to\infty} \|\mathcal{T}\| = \|T_N\|.$$

The proof is given in (Yamamoto *et al.*, 1999). The discretized system T_N is obtained as follows (see Figure 4).

$$T_N = \mathcal{F}_l(G_{dN}, KQ),$$
$$G_{dN}[z] = \left[\begin{array}{c|cc} A_d & B_{1dN} & B_d \\ \hline C_{1dN} & D_{11dN} & D_{12dN} \\ C_2 & 0 & 0 \end{array}\right]$$

where

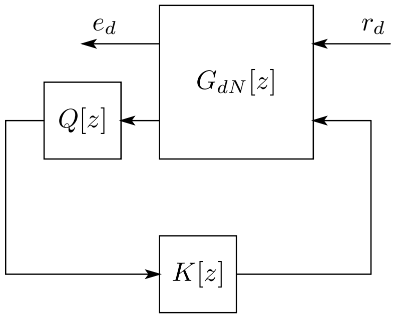

Fig. 5. Standard discrete-time system for controller $K[z]$

$$
\begin{aligned}
A_d &:= e^{A_c h},\\
B_d &:= \int_0^h e^{A_c t} B_2 dt,\\
B_{1dN} &:= \left[A_{dN}^{N-1} B_{1N} \ A_{dN}^{N-2} B_{1N} \ \ldots \ B_{1N} \right],\\
C_{1dN} &:= \begin{bmatrix} C_1 \\ C_1 A_{dN} \\ \vdots \\ C_1 A_{dN}^{N-1} \end{bmatrix},\\
D_{11dN} &:= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ C_1 B_{1N} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_1 A_{dN}^{N-2} B_{1N} & C_1 A_{dN}^{N-3} B_{1N} & \ldots & 0 \end{bmatrix},\\
D_{12dN} &:= \begin{bmatrix} 0 \\ C_1 B_{2N} \\ \vdots \\ \sum_{i=0}^{N-2} C_1 A_{dN}^i B_{2N} \end{bmatrix},\\
A_{dN} &:= e^{A_c h/N},\\
B_{1N} &:= \int_0^{h/N} e^{A_c t} B_1 dt,\\
B_{2N} &:= \int_0^{h/N} e^{A_c t} B_2 dt.
\end{aligned}
$$

In Figure 4, the controller consists of $Q[z]$ and $K[z]$. Our problem is therefore a constrained controller design, which is difficult to solve. We have assumed that $K[z]$ is proper, and hence we shift $Q[z]$ to the upper block as shown in Figure 5, and we find $K[z]$ which is proper and linear time-invariant. As a consequence, we can obtain the optimal controller via a standard discrete-time H^∞ control.

4. DESIGN EXAMPLES

In this section, we present a numerical example to illustrate the effectiveness of our method. We compare our method with a conventional one, that is, a discrete-time LQR design (Nakano *et al.*, 1989).

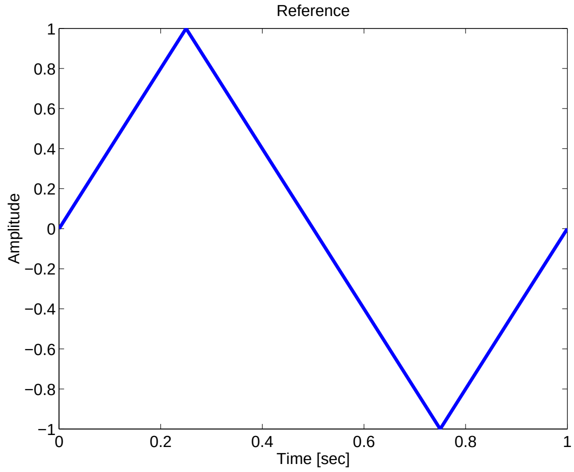

Fig. 6. Reference signal

Let the plant $P(s)$, the anti-aliasing filter $F(s)$ and the weight $W(s)$ be as

$$
\begin{aligned}
P(s) &= \frac{1}{(s+1)(3s+1)},\\
F(s) &= \frac{20}{0.01s+1},\\
W(s) &= \frac{1}{2s+1},
\end{aligned}
$$

respectively. We take the periodic reference signal r as follows:

$$
r(t) = \begin{cases} 4t, & 0 \le t < 1/4, \\ -4t+2, & 1/4 \le t < 3/4, \\ 4t-4, & 3/4 \le t < 1 \end{cases}
$$

of period $T = 1.0$ [sec] (see Figure 6). The sampling period is chosen to be $h = 0.05$ [sec]. Thus, there are $M = 20$ sampling instants in each period of the reference. We employ the fast-sampling/fast-hold method with the discretizing rate $N = 4$.

Figure 7 illustrates transient response of the output $y(t)$ $(0 \le t \le 10)$ of the system designed via sampled-data H^∞ design and the conventional one. It is obvious that the proposed design shows a better transient response compared to the conventional one. Figure 8 shows the steady state response $(20 \le t \le 22)$. We can see that the conventional control shows intersample ripples, while our control attenuates the ripples better. To see the difference precisely, we depict the steady state error in Figure 9, which proves that the proposed method effectively attenuates the intersample ripples, and also shows a better tracking performance in high-frequency.

5. CONCLUSION

We have proposed a sampled-data H^∞ design for attenuating intersample ripples in sampled-data repetitive control. We demonstrate the ef-

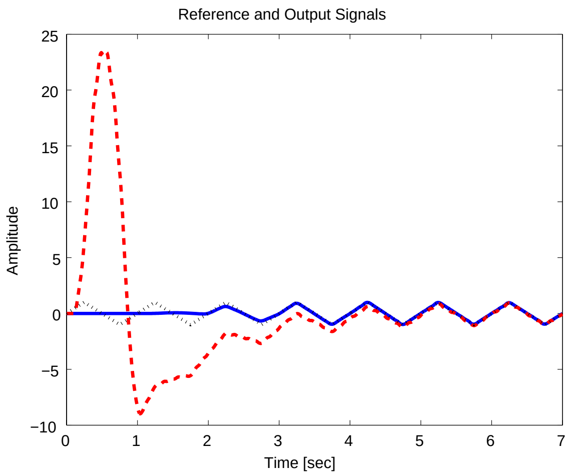

Fig. 7. Transient response: reference (dot), proposed (solid) and conventional (dash)

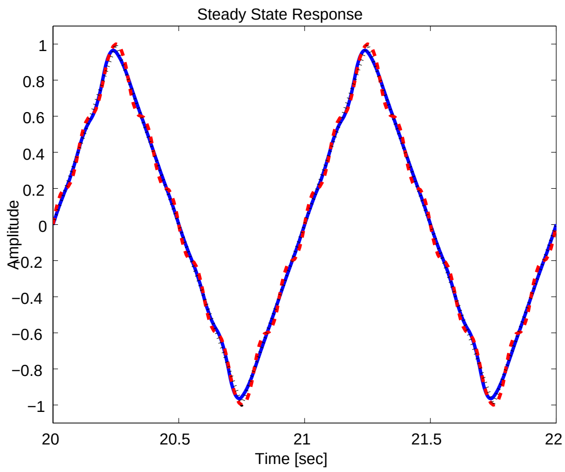

Fig. 8. Steady state response: reference (dot), proposed (solid) and conventional (dash)

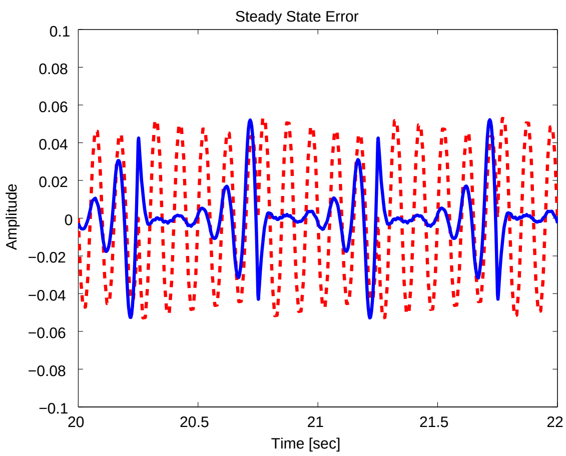

Fig. 9. Steady state error: proposed (solid) and conventional (dash)

fectiveness of our method by comparing it with an existing method.

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