Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Stability of Signal Reconstruction Filters via Cardinal Exponential Splines

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Abstract: There is a new trend in digital signal processing. It is gradually recognized that while the processing is done in the digital domain, its performance must be measured in the analog domain. This framework was proposed by the present authors and co-workers, and also recently proposed by Unser and his co-workers. While our approach relies on modern sampled-data control theory which minimizes an analog H^∞ performance criterion, Unser independently proposed an oblique projection method. This paper examines their method, and shows that their method often leads to instability of designed filters. A comparison with the sampled-data method is made, along with some design examples, which shows the advantage of the sampled-data method.

1. INTRODUCTION

Expanding signals in terms of prespecified basis functions is a fundamental problem in signal processing. Fourier and wavelet analysis are typical examples. In so doing, it is required that such an expansion gives a faithful result, i.e., the error is either zero or very small, and that it is effective in the sense that one needs only a small number of terms to attain sufficient accuracy.

Such an expansion is performed on an a priori given set of data on signals. In the digital context, such data are likely to be given in the form of sampled values of a target signal. Typically, one is given a set of uniformly sampled signal values $\{f(nh)\}_{n=-\infty}^{\infty}$, with sampling period h, and then compute a series function expansion as desired. We do not therefore know the intersample values of the signal and hence we have only partial information of the original continuous-time signal $f(t)$. This loss of information should be in some way recovered to obtain a desirable expansion.

A usual, and now quite standard, approach is to rely on the sampling theorem (Shannon (1949); Zayed (1996)), assuming that the original signal does not contain any frequency components beyond the Nyquist frequency π/h, which is half of the sampling frequency $2\pi/h$.

In examining the hypotheses underlying the sampling theorem, we however notice that there can be certain points that can be improved. For example, it is much more natural to assume that the class of signals obeys a certain frequency decay curve, extending even beyond the Nyquist frequency. It is then desirable to design a digital (discrete-time) filter that optimally reconstructs the original analog signals. Due to the nature of analog characteristics, it is plausible that such an optimal filter gives a result different from a more familiar and rather standard perfect reconstruction filters (Vaidyanathan (2000)) based on the discrete-time performance measures.

Thus our desirable target is to design a digital (discrete-time) filter that optimizes an analog performance index for analog signals. This philosophy and framework were proposed by the present authors in, e.g., Khargonekar and Yamamoto (1996); Yamamoto et al. (2000). To this end, we have shown that modern sampled-data control theory can be very effectively used for signal processing. This is more than natural since this new theory made it possible to design a digital controller that optimally controls an analog performance of a continuous-time plant. This new theory was developed in the 90's and represents a fundamental advance in the design philosophy of sampled-data control theory which was previously based on only discrete-time performance measured at sampled points.

This new theory is adopted to digital signal processing in Khargonekar and Yamamoto (1996). While there are some technical modifications necessary for this, this is basically in the scope of sampled-data theory, and now effectively solvable. Due to the nature of analog characteristics, this new theory can optimize the intersample responses of the processed signals. This is in marked contrast to the conventional philosophy of digital signal processing.

★ This work was supported in part by the JSPS Grant-in-Aid for Scientific Research (B) No. 18360203, and also by Grant-in-Aid for Exploratory Research No. 17656138.

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

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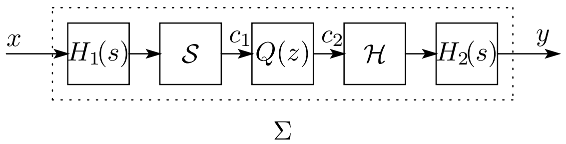

Fig. 1. Sampled-data signal reconstruction

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In particular, this theory can optimize the analog H^∞ performance index, which means the maximum of the frequency response gain. Hence it can control the high-frequency intersample behavior even beyond the Nyquist frequency.

A very similar philosophy is also presented and proposed recently by Unser and co-workers (Unser (2005)). To come up with a solution method, they introduced a new type of splines called cardinal exponential B-splines (Unser and Blu (2005)). This consists of B-spline like functions that are exponentials between sampled points. Assuming that the original signals are linear combinations of exponentials, it is natural to conceive that such signals are expressible as combinations of such piecewise exponential basis functions. The motto is "Think analog, act digital" in Unser (2005) which is exactly the same as the one in sampled-data signal processing theory since Khargonekar and Yamamoto (1996).

Toward a solution of such an expansion, Unser (2005) proposed a method based on oblique projections. The performance is a square norm. They gave a system solution to this problem.

This paper intends to closely examine the merit of Unser's method. In particular, we focus our attention to the stability of designed filters. In fact, Unser (2005) claims that their method always gives rise to a stable filter. We will examine this closely, and actually *this claim is indeed false*, that is, *the proposed method in Unser (2005) can, in general, lead to unstable filters*. This is not coincidental, but is closely connected to the limiting zeros of discretized continuous-time transfer functions, and it shows that Unser (2005) neglects analog characteristics described by plants with order more than 1.

The paper is organized as follows: We start with examining Unser's method, and then give a detailed stability analysis. We will show the relation with Astrom's fundamental result, and show a counterexample in the stability analysis.

2. SIGNAL RECONSTRUCTION BY CARDINAL EXPONENTIAL SPLINES

2.1 Perfect reconstruction by oblique projection

We state and review the signal reconstruction framework by cardinal exponential B-splines (Unser (2005)). Consider the system shown in Fig. 1. Hereafter we assume, without loss of generality, that our sampling period is 1.

Let $x(t) \in L^2$ denote an exogenous signal. Then it is filtered through an analog low-pass filter $H_1(s)$, becoming semi-bandlimited following the characteristics of $H_1(s)$. It is further sampled by the ideal sampler

$$\mathcal{S} : H_1L^2 \ni u \mapsto v \in \ell^2, \quad v[k] := u(k), \quad k = 0, 1, 2, \ldots,$$

and becomes a digital signal $c_1[k]$. We here note that if the low-pass filter $H_1(s)$ satisfies a decay estimate

$$|H_1(j\omega)| \le M(1 + |\omega|^2)^{-\alpha/2}$$

for some $M > 0$ and $\alpha > 1/2$, then for every function u in H_1L^2, the resulting sampled sequence v belongs to ℓ^2 (see, e.g., Kannai and Weiss (1993)). This condition is satisfied, for example, for low-pass filters with transfer function of relative degree greater than or equal to 1. Then the digital filter $Q(z)$, to be designed below, produces another digital signal $c_2[k]$, which is converted back to an analog signal by the zero-order hold $\mathcal{H}$ defined by

$$\mathcal{H} : \ell^2 \ni v \mapsto u \in L^2, \quad u(t) := \sum_{k=0}^{\infty} \beta_0(t-k)v[k],$$

where $\beta_0(t)$ is the hold function

$$\beta_0(t) := \begin{cases} 1, & t \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

The output of $\mathcal{H}$ is then filtered through an analog filter $H_2(s)$ to obtain the final objective analog signal $y(t)$. The signal reconstruction problem is to find a digital filter $Q(z)$ which makes the output y as close to the input x as possible.

Let $h_1(t)$ and $h_2(t)$ denote the impulse responses of $H_1(s)$ and $H_2(s)$, respectively. Then, the digital signal $c_1[k]$ is represented as follows.

$$\begin{aligned} c_1[k] &= \mathcal{S}(h_1 * x)[k] \\ &= \int_0^\infty x(t)h_1(k-t)\mathrm{d}t \\ &= \int_0^\infty x(t)\phi_1(t-k)\mathrm{d}t \qquad (\phi_1(t) := h_1(-t)) \\ &= \langle x(\cdot), \phi_1(\cdot - k)\rangle. \end{aligned}$$

On the other hand, the output analog signal $y(t)$ is represented as follows.

$$y(t) = \sum_{k=0}^{\infty} c_2[k]\phi_2(t-k),$$
$$\phi_2(t) = [\beta_0(\cdot) * h_2(-\cdot)](t).$$

Now define the following spaces:

$$\begin{aligned} V_1 &:= \operatorname{span}\{\phi_1(\cdot - k)\}_{k=0}^{\infty}, \\ V_2 &:= \operatorname{span}\{\phi_2(\cdot - k)\}_{k=0}^{\infty}. \end{aligned} \tag{1}$$

Summarizing, the above setup says the following: The exogenous signal x is filtered through an analog filter $H_1(s)$. This in general limits the bandwidth of the filtered signal, usually in a low-pass characteristic, according to the decay characteristic of $H_1(s)$. The filtered signal is then sampled. The shifted linear span of $\phi_1(\cdot)$ is V_1. Likewise V_2 is precisely the space of outputs produced by the hold device $\mathcal{H}$ and $H_2(s)$. Note, however, that V_1 never represents the totality of signals obtained by filtering $x \in L^2$ by H_1. The latter space is infinite-dimensional, while V_1 is spanned by linear (possibly infinitely many) combinations of shifted $\phi_1(t) = h_1(-t)$.

Suppose, for the moment, that we want to reconstruct given signals in V_1 with those in V_2. That is, we consider the following signal reconstruction problem:

Problem A: Given an arbitrary exogenous signal in $x \in L^2$, find $y \in V_2$ such that y optimally approximates x in the sense of L^2.

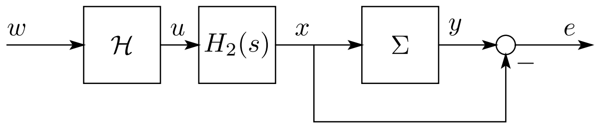

Fig. 2. Error system E_w

In an analogy with the well known projection theorem, Unser (2005) proposed that $x - y$ be orthogonal to V_1. Note that this gives a best approximant if ϕ_1 belongs to V_2. Since ϕ_1 does not necessarily belong to V_2, one needs a modification. Namely, the optimal reconstruction solution proposed in Unser (2005) is characterized by

$$\langle x(\cdot) - y(\cdot), \phi_1(\cdot - k)\rangle = 0, \quad k = 0, 1, 2, \ldots. \tag{2}$$

This idea states that there remains no extra component in the error $x - y$ that can be expanded with elements in V_1. The filter $Q(z)$ corresponds to the oblique projection Unser (2000) of $x(t)$ onto V_2 orthogonal to V_1.

Definition 1. A linear operator $P : L^2 \to V_2$ is an *oblique projection* on V_2 orthogonal to V_1 if the following three conditions hold:

(1) $Px = x, \forall x \in V_2$.
(2) $x - Px \in V_1^\perp, \forall x \in L^2$.
(3) $Px = 0, \forall x \in V_1^\perp$.

We quote the following proposition from Unser (2005):

Proposition 1. The optimal filter, denoted by $Q_{\rm op}$, achieving (2) makes the operator Σ (see Fig. 1) to be the oblique projection on V_2 orthogonal to V_1.

Now consider the error system shown in Fig. 2. In this figure, Σ is the operator from x to y shown in Fig. 1. By E_w we denote the error system given by Fig. 2. We then have the following proposition.

Proposition 2. The optimal filter $Q_{\rm op}$ achieving (2) is also optimal in the sense that

$$Q_{\rm op}(z) = \operatorname*{argmin}_{Q(z)} \|E_w\|_2 = \operatorname*{argmin}_{Q(z)} \|E_w\|_\infty.$$

Proof. Assume that the input w is the discrete-time delta function

$$\delta[k] = \begin{cases} 1, & k = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then $x \in V_2$. Since Σ is an oblique projection,

$$y = \Sigma x = x.$$

That is, the impulse response of the error system E_w is zero.

Next, assume that the input w is in ℓ^2. In this case, x is also in V_2 and $e = y - x = 0$. That is, $Q_{\rm op}(z)$ is the optimal filter to make $\|E_w\|_\infty = 0$. □

In the next subsection, we will show a formula for the filter $Q_{\rm op}(z)$.

2.2 Characterization by system inversion

The formula of the optimal $Q_{\rm op}(z)$ is obtained by the cardinal exponential B-spline method (Unser and Blu (2005); Unser (2005)).

First, let us consider the rational transfer function

$$H_{\boldsymbol{\alpha}}(s) = \frac{(s-\gamma_1)(s-\gamma_2)\cdots(s-\gamma_M)}{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_N)},$$
$$\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_N; \gamma_1, \ldots, \gamma_M).$$

Then the B-spline function corresponding to $H_{\boldsymbol{\alpha}}(s)$ is given by Unser (2005)

$$\hat{\beta}_{\boldsymbol{\alpha}}(s) = \Delta_{\boldsymbol{\alpha}}(s) H_{\boldsymbol{\alpha}}(s),$$

where $\Delta_{\boldsymbol{\alpha}}(s)$ is the *localization operator* (Unser and Blu (2005); Unser (2005)) defined as

$$\Delta_{\boldsymbol{\alpha}}(s) = (1 - e^{\alpha_1} e^{-s})(1 - e^{\alpha_2} e^{-s}) \cdots (1 - e^{\alpha_N} e^{-s}). \tag{3}$$

Note that each $1 - e^{\alpha_i} e^{-s}$ is a truncation operator (Mirkin (2003)), and the support of $\beta_{\boldsymbol{\alpha}}(t)$ (the inverse Laplace transform of $\hat{\beta}_{\boldsymbol{\alpha}}(s)$) is contained in $[0, N]$.

By using this B-spline function, the optimal filter $Q_{\rm op}(z)$ is given as follows. Let the poles and zeros of $H_1(s)$ and $H_2(s)$ be

$$\boldsymbol{\alpha}_1 = (\alpha_{11}, \ldots, \alpha_{1N_1}; \gamma_{11}, \ldots, \gamma_{1M_1}),$$
$$\boldsymbol{\alpha}_2 = (\alpha_{21}, \ldots, \alpha_{2N_2}; \gamma_{21}, \ldots, \gamma_{2M_2}).$$

For brevity, we assume that α_{ij}'s are distinct, but this is not at all necessary. Then the optimal $Q_{\rm op}(z)$ is obtained by the following equation (Unser (2005)):

$$Q_{\rm op}(z) = \frac{\Delta_{\boldsymbol{\alpha}_1}(z)\Delta_{\boldsymbol{\alpha}_2}(z)}{\displaystyle\sum_{k=0}^{N_1+N_2+1} \beta_{(\boldsymbol{\alpha}_1:0:\boldsymbol{\alpha}_2)}(k) z^{-k}}, \tag{4}$$

where $\beta_{(\boldsymbol{\alpha_1}:0:\boldsymbol{\alpha_2})} := \beta_{\boldsymbol{\alpha_1}} * \beta_0 * \beta_{\boldsymbol{\alpha_2}}$, and $\Delta_{\boldsymbol{\alpha_i}}$ $(i = 1, 2)$ is the discretization of the localization operator of $\Delta_{\boldsymbol{\alpha_i}}(s)$, that is,

$$\Delta_{\boldsymbol{\alpha_i}}(z) = \prod_{n=1}^{N_i} (1 - e^{\alpha_{in}} z^{-1}), \quad i = 1, 2. \tag{5}$$

The filter (4) is based on cardinal exponential B-splines, and the realization of this filter is easily executed by the spline calculus (see Unser and Blu (2005); Unser (2005)) [1]. On the other hand, this filter is realized via system inversion.

Theorem 1. The optimal filter $Q_{\rm op}(z)$ in (4) can be equivalently realized as

$$Q_{\rm op}(z) = \frac{1}{H_{12d}(z)}, \tag{6}$$

where $H_{12d}(z)$ is the step-invariant equivalent discretization of $H_1(s)H_2(s)$, that is, if a state-space realization of $H_1(s)H_2(s)$ is given by $\{A, B, C, 0\}$, then

$$H_{12d}(z) = \mathcal{S} H_1(s) H_2(s) \mathcal{H} = \left[\begin{array}{c|c} e^A & \int_0^1 e^{A\tau} B d\tau \\ \hline C & 0 \end{array}\right].$$

Proof. First, we consider the denominator of $Q_{\rm op}(z)$. The Laplace transform of $\beta_{(\boldsymbol{\alpha}_1:0:\boldsymbol{\alpha}_2)}(t)$ is given by

$$\begin{aligned} \hat{\beta}_{(\boldsymbol{\alpha}_1:0:\boldsymbol{\alpha}_2)}(s) &= \mathcal{L}\left[\beta_{(\boldsymbol{\alpha}_1:0:\boldsymbol{\alpha}_2)}\right](s) \\ &= \mathcal{L}\left[\beta_{(\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_2)} * \beta_0\right](s) \\ &= \hat{\beta}_{(\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_2)}(s) \frac{1 - e^{-s}}{s}. \end{aligned}$$

[1] However, this filter is not guaranteed to be stable, as we see in the next section. Of course, if x belongs to V_2 and if there are no errors in computation, the formula still works for unstable filters, but in practice this does not work.

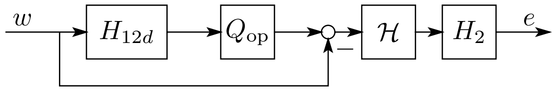

Fig. 3. Block diagram equivalent to Fig. 2

The coefficients $\beta_{(\boldsymbol{\alpha}_1:0:\boldsymbol{\alpha}_2)}(k)$, $k = 0, 1, 2, \dots$ are obtained by sampling the inverse Laplace transform of $\hat{\beta}_{(\boldsymbol{\alpha}_1:0:\boldsymbol{\alpha}_2)}(s)$. Since $(1 - e^{-s})/s$ is the laplace transform of the hold function $\beta_0(t)$ of $\mathcal{H}$, the denominator of $Q_{\rm op}(z)$ is the z-transform of the step-invariant transformation of $\beta_{(\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_2)}$, that is,

$$\mathcal{S}\hat{\beta}_{(\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_2)}(s)\mathcal{H} = \mathcal{S}\Delta_{\boldsymbol{\alpha}_1}(s)\Delta_{\boldsymbol{\alpha}_2}(s)H_1(s)H_2(s)\mathcal{H}.$$

By using the relation $\mathcal{S}e^{-s} = z^{-1}\mathcal{S}$, we have (see (3) and (5))

$$\mathcal{S}\Delta_{\boldsymbol{\alpha}_1}(s)\Delta_{\boldsymbol{\alpha}_2}(s) = \Delta_{\boldsymbol{\alpha}_1}(z)\Delta_{\boldsymbol{\alpha}_2}(z)\mathcal{S}.$$

It follows that

$$\begin{aligned}\mathcal{S}\hat{\beta}_{(\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_2)}(s)\mathcal{H} &= \Delta_{\boldsymbol{\alpha}_1}(z)\Delta_{\boldsymbol{\alpha}_2}(z)\mathcal{S}H_1(s)H_2(s)\mathcal{H}\\ &= \Delta_{\boldsymbol{\alpha}_1}(z)\Delta_{\boldsymbol{\alpha}_2}(z)H_{12d}(z).\end{aligned}$$

Then, since the numerator of $Q_{\rm op}(z)$ is $\Delta_{\boldsymbol{\alpha}_1}(z)\Delta_{\boldsymbol{\alpha}_2}(z)$, we conclude that

$$Q_{\rm op}(z) = \frac{\Delta_{\boldsymbol{\alpha}_1}(z)\Delta_{\boldsymbol{\alpha}_2}(z)}{\mathcal{S}\hat{\beta}_{(\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_2)}\mathcal{H}} = \frac{1}{H_{12d}(z)}.$$

□

We have shown in the previous subsection that the optimal filter $Q_{\rm op}(z)$ makes the transfer function of the error system E_w (Fig. 2) zero. On the other hand, the block diagram in Fig. 2 is equivalently transformed to Fig. 3. From this block diagram, we easily see that the filter (6) makes the transfer function E_w zero.

3. STABILITY OF CARDINAL EXPONENTIAL B-SPLINE RECONSTRUCTION

It is claimed in Unser (2005) that the filter $Q_{\rm op}(z)$ is stable if $\cos(\theta_{12}) \neq 0$, where θ_{12} is the angle between V_1 and V_2.

However, this is not necessarily true according to the following well-known result of limiting zeros by Åstöm et al. (1984):

Fact 3. For every continuous-time system with relative degree strictly greater than 2, its step-invariant-discretized system always possess an unstable zero provided that the samping time is sufficiently small.

Even if the sampling time (= 1 in our present normalization) is not small compared to the time-constants of H_1 and H_2, the discretized system $H_{12d}(z)$ may still have unstable zeros.

For example, consider

$$H_1(s) = \frac{1}{s+1}, \quad H_2(s) = \frac{1}{(s+1.5)(s+2)}.$$

An easy calculation via MATLAB shows $\cos(\theta_{12}) = 0.4863 \neq 0$. The zeros of the discretized system $H_{12d}(z)$ are

$$\{0, -1.28549, -0.0816767\},$$

and hence the optimal filter

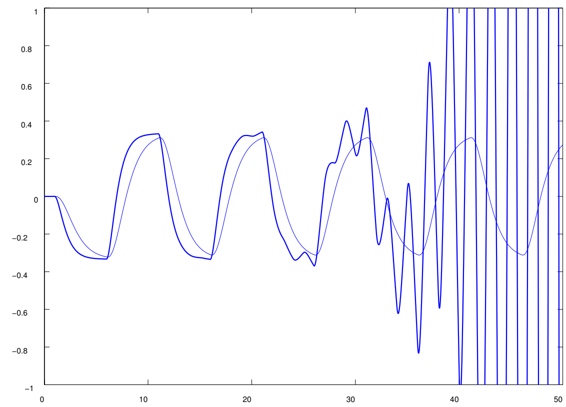

Fig. 4. Time response of Unser's reconstruction system

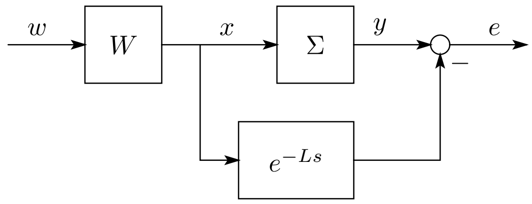

Fig. 5. Error system E_w for sampled-data delayed signal reconstruction

$$\begin{aligned}Q_{\rm op}(z) &= \frac{1}{H_{12d}(z)}\\ &= \frac{z^3 - 0.7263z^2 + 0.1621z - 0.01111}{0.05725z^3 + 0.07827z^2 + 0.006011z}\end{aligned}$$

is unstable. This $Q_{\rm op}$ agrees exactly with the one obtained via oblique projections; see, e.g., `http://bigwww.epfl.ch/demo/Esplines/`.
Fig. 4 shows a time response of the reconstruction system. The output clearly diverges.

4. ROBUST RECONSTRUCTION VIA SAMPLED-DATA H^∞ OPTIMIZATION

In the previous section, the reconstruction system proposed in Unser (2005) is not necessarily stable. Moreover, Theorem 1 shows that this system does not take the intersample behavior into account.

In contrast, we have dealt in Khargonekar and Yamamoto (1996); Yamamoto et al. (2000) with the same problem via sampled-data H^∞ control theory.

Consider the block diagram in Fig. 5 (cf. the error system in Fig. 2). In this diagram, the input w is in L^2, and W is a model of the input analog signal[2]. We assume that W is a continuous-time linear time-invariant system with rational transfer function $W(s)$. In the upper portion of the diagram, input x is discretized, processed in discrete-time, and then converted back to a continuous-time signal y (see Fig. 1). In the lower portion, the input analog signal is delayed and forwarded to the adding point. The

[2] In the case of the cardinal exponential B-spline reconstruction, the input model W is $H_2(s)\mathcal{H}$. The sampling theorem by Shannon assumes that W is an ideal lowpass filter with cutoff frequency $\omega_c < \pi$ (Nyquist frequency).

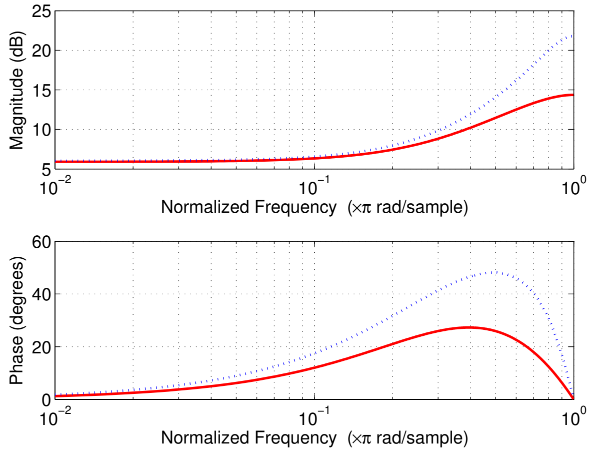

Fig. 6. Bode plot of reconstruction filters $Q_{\mathrm{op}}(z)$: cardinal exponential B-spline method (dot) sampled-data H^∞ optimization (solid)

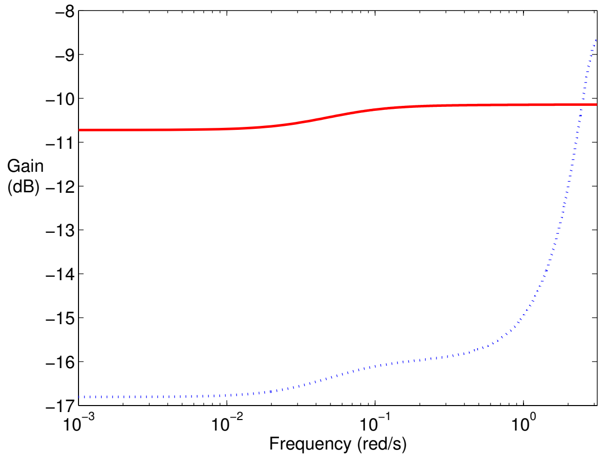

Fig. 7. Frequency response (gain) of error system E_w: exponential B-spline method (dot) sampled-data H^∞ optimization (solid)

objective is to attenuate the reconstruction error e with respect to the H^∞ norm of the error system E_w. The optimal filter is then obtained by modern sampled-data control theory, and has the following advantages:

(1) the optimal filter is always stable;
(2) the design takes the inter-sample behavior into account;
(3) the system is robust against the uncertainty of W;
(4) the optimal FIR filter is also obtainable; via Linear Matrix Inequalities (LMI).

5. NUMERICAL EXAMPLES

Here we design the reconstruction filter $Q_{\mathrm{op}}(z)$ by

- cardinal exponential B-spline reconstruction (Unser (2005)), and
- sampled-data H^∞ optimization (Khargonekar and Yamamoto (1996); Yamamoto et al. (2000)).

Example 1. The analog filters are the same as those given in Unser (2005)

$$H_1(s) = \frac{1}{s+1}, \quad H_2(s) = \frac{1}{s+2}.$$

The reconstruction delay is set to be $m = 1$ to be consistent with Unser (2005). While the setup of Unser (2005) appears quite similar to the sampled-data setup (Fig. 5), we do need yet another transfer function $W(s)$ to make the sampled-data design work [3]. We here take $W(s)$ as

$$W(s) = \frac{1}{s+0.05}.$$

Fig. 6 shows the optimal filter $Q_{\mathrm{op}}(z)$ designed by the cardinal exponential B-spline method and sampled-data H^∞ optimization. Fig. 7 shows the frequency response of the error system E_w shown in Fig. 5. Sampled-data H^∞ optimized system shows almost a flat characteristic. On the other hand, the exponential B-spline reconstruction system shows much larger error in high frequencies. Figures 8 and 9 show the time responses against the square-wave by the exponential B-spline reconstruction system, and by the sampled-data designed filter. The former shows a large amount of ripples around the edges.

[3] This W cannot be absorbed to $H_1(s)$ in Unser's framework, since what is to be reconstructed is the weighted w, namely x, while Unser (2005) attempts to track unweighted input w, but with the expense of stability as we already pointed out.

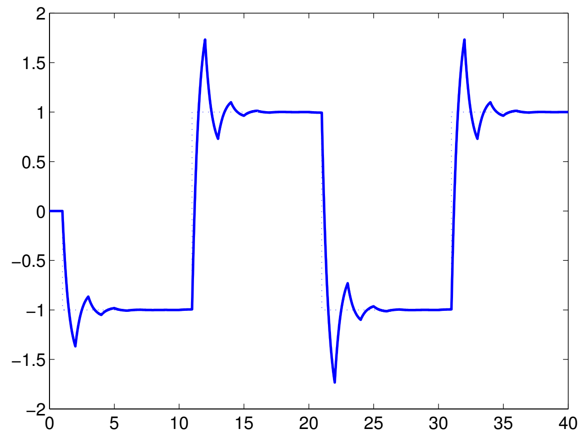

Fig. 8. Time response (exponential B-spline reconstruction)

In the above example, the sampling time 1 is close to the time constants of H_1 and H_2. This makes the advantage of the sampled-data design a little ambiguous. In the next example, where the time constant of H_2 is large compared to the sampling period, there is a more obvious distinction.

Example 2. We take $H_2(s) = 1/(s+0.05)$. H_1 and W are the same as before. The zeros of the step-invariant-discretization H_{12d} of $H_1(s)H_2(s)$ are $\{0, -0.7063\}$, and hence the filter based on exponential B-splines is stable. Indeed, we have $Q_{\mathrm{op}}(z) = (z^2 - 1.319z + 0.3499)/(0.3614z^2 + 0.2552z)$. This filter leads to one step delay ($m = 1$). On the other hand, our design is free from the limit of the delay, and hence we design the optimal filter with $m = 1$ and $m = 4$.

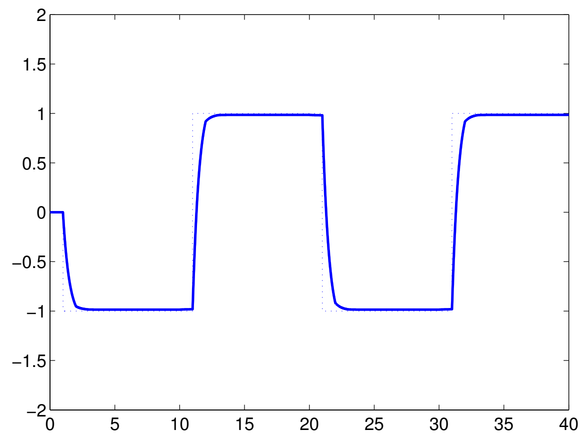

Fig. 9. Time response (sampled-data H^∞ optimization)

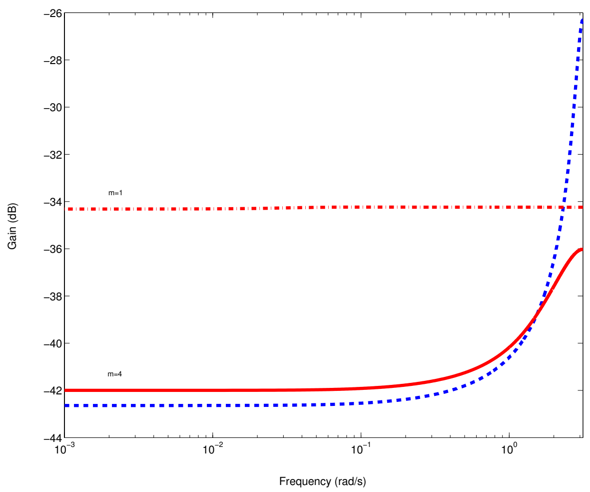

Fig. 10. Frequency response (gain) of error system E_w: exponential B-spline method (dash) sampled-data H^∞ optimization with $m = 1$ (dash-dot) and $m = 4$ (solid)

Fig. 10 shows the frequency response of the error system E_w shown in Fig. 5. Sampled-data H^∞ optimized system ($m = 1$) shows almost a flat characteristic. On the other hand, the exponential B-spline reconstruction system shows much larger error in high frequencies. Note also that as we increase the delay step m, the error is substantially reduced in the sampled-data design. While we omit their time ersponses due to the lack of space, the design via cardinal exponential B-splines shows much larger ripples around the edges for square-wave responses.

6. CONCLUDING REMARKS

We have shown that the reconstruction filter proposed in Unser (2005) is the inverse of the step-invariant discretization of the pertinent analog filters. Hence this reconstruction system does not take the intersample behavior of continuous-time signals into account. We have also shown that the obtained filter is not necessarily stable in spite of the claim in Unser (2005). In contrast, the sampled-data H^∞ optimal reconstruction proposed in Khargonekar and Yamamoto (1996); Yamamoto et al. (2000) is always stable and can take the intersample behavior into accout. The numerical examples show the advantages of the sampled-data H^∞ design. The sampled-data design has also been studied extensively for sample-rate conversion (Ishii et al. (1999); Nagahara and Yamamoto (2000); Yamamoto (2006b)), fractional delay filters (Nagahara and Yamamoto (2003)), digital image processing (Kakemizu et al. (2005)) etc. For details, the readers are referred to these articles and references therein.

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