OPTIMAL NOISE SHAPING IN $\Delta\Sigma$ MODULATORS VIA GENERALIZED KYP LEMMA

Masaaki Nagahara

Kyoto University
Graduate School of Informatics
Kyoto 606-8501, JAPAN
nagahara@ieee.org

Yutaka Yamamoto

Kyoto University
Graduate School of Informatics
Kyoto 606-8501, JAPAN
yy@i.kyoto-u.ac.jp

ABSTRACT

In this paper, we propose a new design method of $\Delta\Sigma$ modulators. First, we analyze the stability of $\Delta\Sigma$ modulators. We show a parametric representation of all stabilizing loop filters for a linearized model, and then an analysis of the nonlinear stability is discussed. Next, by using the parameterization of loop filters, we propose an optimal design to shape the frequency response of the noise transfer function (NTF). Generalized KYP (Kalman-Yakubovic-Popov) lemma is used to reduce our optimization to a linear matrix inequality.

Index Terms— $\Delta\Sigma$ modulation, quantization, H^∞ optimization, generalized KYP lemma

1. INTRODUCTION

$\Delta\Sigma$ *modulators* [1] are widely used in AD (Analog-to-Digital) and DA (Digital-to-Analog) converters, in which high performance can be obtained with coarse quantizers.

A fundamental issue in designing $\Delta\Sigma$ modulators is *noise shaping* in the frequency domain [1]. A usual solution to this is to insert accumulator(s) in the feedback loop to attenuate the gain of the noise transfer function (NTF) in low frequencies. This methodology looks like PID (Proportional-Integral-Derivative) control [2], in which the performance of the designed system depends on the amount of experiences of the designer. That is, the conventional design is of an ad hoc nature.

Let us consider a general $\Delta\Sigma$ modulator shown in Fig. 1. In this modulator, Q is a quantizer and $H = [H_1, H_2]$ is a linear filter with 2 inputs and 1 output. The filter H_1 shapes the signal transfer function (STF) from the input u to the output y to be 1 in the frequency band of interest. On the other hand, the filter H_2 eliminates the in-band quantization noise by shaping the NTF.

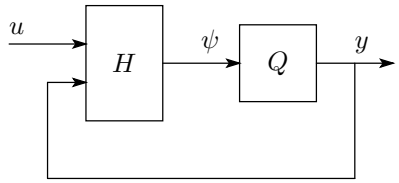

Fig. 1. $\Delta\Sigma$ modulator

To shape *optimally* the NTF in the frequency band of interest, say $[0, \Omega]$, we minimize the maximum of the gain of the NTF in $[0, \Omega]$. This is related to a minimax optimization (or an H^∞ one). We have proposed an H^∞ optimization in [3], in which we have to choose a suitable weighting function to obtain a good performance. On the other hand, we propose in this article more useful method with no weighting function, by generalized Kalman-Yakubovic-Popov (KYP) lemma [4]. Then the optimization can be reduced to one with a linear matrix inequality (LMI). The idea to apply generalized KYP lemma to $\Delta\Sigma$ modulator design is also proposed in [5], in which they assume one-bit quantizer for Q and optimize the average power of the reconstruction error in low frequencies. In contrast to this approach, our optimization is for quantization noise shaping, which is more familiar to engineers and researchers in this area. Moreover we assume more general quantizers and under these quantizers we also analyze the stability of $\Delta\Sigma$ modulators.

2. FREQUENCY SHAPING IN $\Delta\Sigma$ MODULATORS

In this section, we discuss a role of the linear system $H = [H_1, H_2]$ (the loop filter) in the $\Delta\Sigma$ modulator shown in Fig. 1. To make the analysis easy, we use the linearized model shown in Fig. 2. By using this model, the input-output equation of the modulator in Fig. 2 is given by $y = T_{\text{STF}}\, u + T_{\text{NTF}}\, n$, where

$$T_{\text{STF}}(z) := \frac{H_1(z)}{1 - H_2(z)}, \quad T_{\text{NTF}}(z) := \frac{1}{1 - H_2(z)}.$$

We call $T_{\text{STF}}(z)$ and $T_{\text{NTF}}(z)$ the signal transfer function (STF) and the noise transfer function (NTF), respectively. For a conventional first order $\Delta\Sigma$ modulator, the loop filters

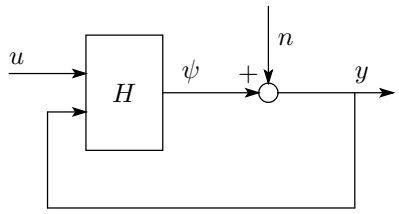

Fig. 2. Linearized model for $\Delta\Sigma$ modulator

are given by

$$H_1(z) = \frac{z}{z-1}, \quad , H_2(z) = -\frac{1}{z-1}. \tag{1}$$

Then we have $y = u + (1-z^{-1})n$, and $T_{\text{NTF}}(z) = 1 - z^{-1}$ is a highpass filter. This is a reason for setting the accumulator $1/(z-1)$ in the loop. By this, the quantization noise is modulated to high frequencies, and if the input signal u contains few high frequency components, we can separate the noise n from the output signal y by an appropriate lowpass filter. To sum up, the loop filter $H(z)$ plays a noise-shaping role in $\Delta\Sigma$ modulators.

Before discussing an optimal design of the loop filter, we discuss the stability of $\Delta\Sigma$ modulators in the next section.

3. CHARACTERIZATION OF LOOP FILTERS

In this section, we first characterize all $H(z)$'s which stabilize the linearized model shown in Fig. 2, and then we consider the stability of the nonlinear system in Fig. 1. A necessary condition that a $\Delta\Sigma$ modulator is stable is that its linearized model is internally stable. The converse is generally not true, that is, even if the linearized model is stable, the nonlinear system in Fig. 1 can be unstable.

We first characterize the filter $H(z)$ which internally stabilizes the linearized feedback system. All stabilizing filters are characterized as follows [3].

Lemma 1. *The linearized feedback system in Fig. 2 is well-posed and internally stable if and only if*

$$\begin{aligned} &H_1(z) = \frac{R_1(z)}{1+R_2(z)}, \quad H_2(z) = \frac{R_2(z)}{1+R_2(z)}, \\ &R_1(z) \in \mathcal{S}, \quad R_2(z) \in \mathcal{S}', \end{aligned} \tag{2}$$

where $\mathcal{S}$ is the set of all stable, causal, real-rational transfer functions, and $\mathcal{S}' := \{R \in \mathcal{S} : R \text{ is strictly causal}\}$.

By using these parameters $R_1 \in \mathcal{S}$ and $R_2 \in \mathcal{S}'$, we have

$$T_{\text{STF}}(z) = R_1(z), \quad T_{\text{NTF}}(z) = 1 + R_2(z),$$

and the input/output equation of the system in Fig. 2 is given by

$$y = R_1 u + (1 + R_2)n. \tag{3}$$

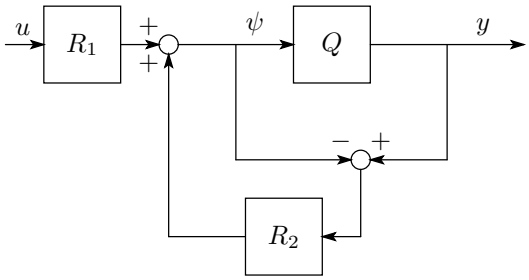

Fig. 3. Structure of $\Delta\Sigma$ modulator with design parameters $R_1 \in \mathcal{S}$ and $R_2 \in \mathcal{S}'$

For example, the conventional modulator in (1) has $R_1(z) = 1 \in \mathcal{S}$ and $R_2(z) = -z^{-1} \in \mathcal{S}'$.

By (3), the structure of the $\Delta\Sigma$ modulator with the design parameters $R_1 \in \mathcal{S}$ and $R_2 \in \mathcal{S}'$ is shown in Fig. 3. By this block diagram, we can interpret the filter R_1 as a pre-filter to shape the frequency response of the input signal, and R_2 as a feedback gain for the quantization noise $Q\psi - \psi$.

Next, we discuss the stability of nonlinear $\Delta\Sigma$ modulators in Fig. 1. We here assume the following.

Assumption 1.
1. *The linearized model is stable. That is, the filter $H(z) = [H_1(z), H_2(z)]$ satisfies (2).*
2. *There exist real numbers $M > 0$ and $\delta > 0$ such that if $|\psi| \le M$ then $|Q\psi - \psi| \le \delta$.*

For example, the uniform quantizer with step size 2δ and no-overload input range $[-M, M]$ (see section 2.1 of [1]) satisfies the second assumption.

Under these assumptions, we have the following lemma.

Lemma 2. *Assume Assumption 1. If $|\psi(0)| \le M$ and $\|r_1\|_1\|u\|_\infty + \|r_2\|_1\delta \le M$ then for all $k \ge 0$, we have $|(Q\psi)(k) - \psi(k)| \le \delta$ and $|\psi(k)| \le M$, where r_1 and r_2 are respectively the impulse responses of R_1 and R_2, and $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are respectively the ℓ^1 and ℓ^∞ norm of signals.*

Proof. The proof uses the technique discussed in section 4.2.2 of [1]. We omit the proof. □

This lemma gives a sufficient condition so that the amplitude of the input ψ of the quantizer Q is always less than the saturation level M of Q. This property is said to be of no-overload, and if this is satisfied a $\Delta\Sigma$ modulator is said to be stable [1]. We here consider the stability more precisely. We introduce a state space model of the $\Delta\Sigma$ modulator shown in Fig. 1, and analyze the stability in the state space.

Let state space equations of the system in Fig. 1 be as follows.

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_n n(k), \\ \psi(k) &= Cx(k) + Du(k), \\ n(k) &= (Q\psi - \psi)(k). \end{aligned}$$

Consider the ideal state $x_I(k)$, which is the state when there is no quantization, that is, when Q is identity. Define the state-space error $e := x - x_I$. Then, we have the following theorem.

Theorem 1. *Suppose that the $\Delta\Sigma$ modulator in Fig. 1 satisfies Assumption 1. If $|\psi(0)| \le M$ and*

$$\|r_1\|_1\|u\|_\infty + \|r_2\|_1\delta \le M \tag{4}$$

then for all $k \ge 0$,

$$|e(k)| \le \beta_k := \delta \sum_{i=0}^{k-1} \|A^i B_n\|.$$

Proof. By using the inequality $|Ax| \le \|A\||x|$ for a vector x and a matrix A, and by Lemma 2, the theorem can be easily proved. We omit the proof. □

By this theorem, when a $\Delta\Sigma$ modulator satisfies the condition in Theorem 1, the quantization error $e(k)$ in the state space is bounded by β_k. This bound is finite for all $k \ge 0$ and $\beta_\infty := \lim_{k\to\infty} \beta_k$ is also finite, since the matrix A is Schur-stable [1] by Assumption 1.1. As a result, the state $x(k)$ is also bounded, and we can conclude that the system is stable in a weak sense (i.e., bounded but not guaranteed to converge to zero).

Assuming that $\|r_1\|_1 = 1$, the condition (4) can be described by the H^∞ norm of R_2, that is, $\|R_2\|_\infty \le C$ where $C > 0$ is a constant (see [3]). This means that to guarantee the stability one cannot arbitrarily increase the feedback gain $\|R_2\|_\infty$. This property is due to the nonlinear nature of $\Delta\Sigma$ modulators.

4. OPTIMAL LOOP FILTER DESIGN VIA GENERALIZED KYP LEMMA

In this section, we propose an optimal design of the loop filter $H(z)$ by using the parameterization in Lemma 1. For simplicity, we assume $R_1(z) = 1$. This means that the STF is assumed to be allpass. Then our problem is formulated as follows.

Problem 1. *Given Ω $(0 < \Omega < \pi)$ and $\gamma > 0$, find $R_2(z) \in \mathcal{S}'$ which satisfies*

$$\sup_{\omega\in[0,\Omega]} |T_{\text{NTF}}(e^{j\omega})| < \gamma. \tag{5}$$

In implementation, finite impulse response (FIR) filters are often preferred, and hence we assume that $R_2(z)$ is FIR, that is, we set

$$R_2(z) = \sum_{k=0}^{N} \alpha_k z^{-k}, \quad \alpha_0 = 0.$$

[1] A matrix A is Schur-stable iff $\rho(A) < 1$ where $\rho(A)$ is the spectral radius of A.

Note that $R_2(z)$ is always in $\mathcal{S}'$. We then introduce state-space matrices $\{A, B, C(\alpha)\}$, such that $R_2(z) = C(\alpha)(zI - A)^{-1}B$, where $\alpha = \begin{bmatrix}\alpha_0 & \alpha_1 & \dots & \alpha_N\end{bmatrix}$,

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $C(\alpha) = [\alpha_N, \alpha_{N-1}, \dots, \alpha_1]$. Then the inequality (5) can be described as linear matrix inequalities (LMI) by using the generalized KYP lemma [4].

Theorem 2. *The inequality (5) holds if and only if there exist symmetric matrices $Q > 0$ and P such that the LMI (6) (printed in the top of the next page) holds.*

By Theorem 2, the optimal coefficients $\alpha_1, \dots \alpha_N$ can be obtained efficiently by standard optimization softwares, e.g., MATLAB (See [6]).

5. DESIGN EXAMPLE

In this section, we show examples of designing $\Delta\Sigma$ modulators by the proposed method.

We here design the filter $R_2(z)$ which is an FIR filter with 12 taps, and set $R_1(z) = 1$. The cut-off frequency Ω is $3\pi/32$. The NTF $1+R_2(z)$ is designed to have a zero at $z = 1$ to attenuate DC noise most, and also to satisfy the stability condition $\|T_{\text{NTF}}\|_\infty < 1.5$ (these constraints can be described as linear matrix equality and inequality, see [3]). By this optimization, we obtain the minimum value of $\gamma = 6.48 \times 10^{-2}$ (-23.8 [dB]). Fig. 4 shows T_{NTF}'s by the proposed method and the first order $\Delta\Sigma$ modulator. The T_{NTF} of our design shows a lower gain in the low frequency and a higher gain in the high frequency. The frequency response in Fig. 4 is that of the linearized system shown in Fig. 2. To see the nonlinear effect in the quantizer, we simulate responses against sinusoidal waves with various frequencies. The reconstruction filter after the $\Delta\Sigma$ modulator is chosen to be H^∞ optimal one proposed in [3]. Fig. 5 shows NSR (Noise-to-Signal Ratio) against sinusoidal waves. The NSR shows that our $\Delta\Sigma$ modulator shows a better response than the conventional one in all frequencies. Fig. 6 and Fig. 7 shows outputs respectively of proposed and conventional $\Delta\Sigma$ converters against a sinusoidal wave.

6. CONCLUSION

In this paper, we have propose a new design method of $\Delta\Sigma$ modulators. We have characterized the all stabilizing loop filters for linearized model, and analyzed the stability of nonlinear $\Delta\Sigma$ modulators. Then we have formulated our problem of noise shaping in the frequency domain. By using generalized

$$\begin{bmatrix} A^\top PA + QA + A^\top Q - P - 2Q\cos\Omega & A^\top PB + QB & C(\alpha)^\top \\ (A^\top PB + QB)^\top & B^\top PB - \gamma^2 & 1 \\ C(\alpha) & 1 & -1 \end{bmatrix} < 0. \quad (6)$$

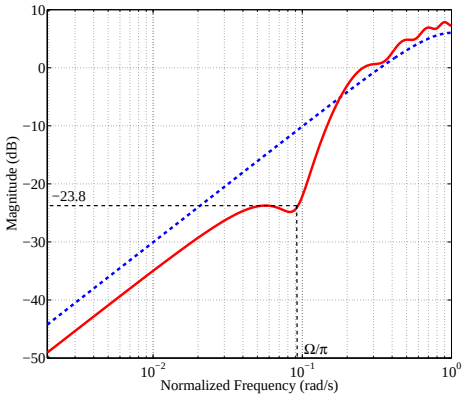

Fig. 4. Frequency response of T_{NTF}: proposed (solid line) and conventional (dash)

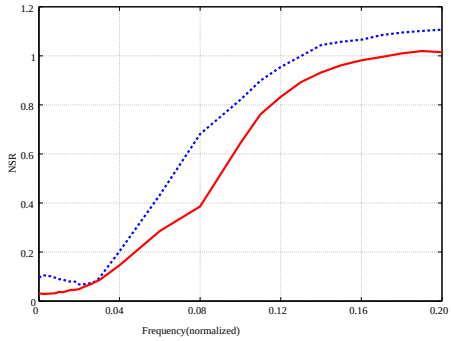

Fig. 5. NSR against sinusoidal waves: proposed (solid) and conventional (dash)

KYP lemma, our design is reducible to an LMI optimization. Design examples have shown efficiency of our method.

7. REFERENCES

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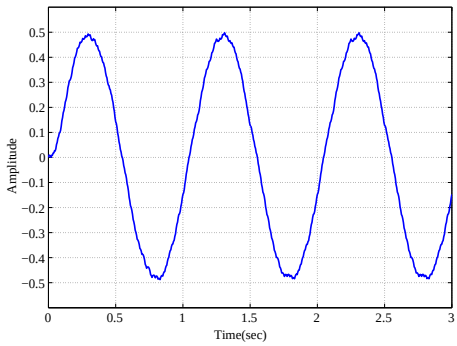

Fig. 6. Output against sinusoidal wave (proposed)

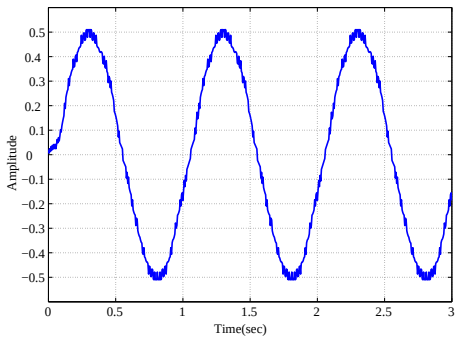

Fig. 7. Output against sinusoidal wave (conventional)