CAUSAL SPLINE INTERPOLATION BY H^∞ OPTIMIZATION

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ABSTRACT

Spline interpolation systems generally contain non-causal filters, and hence such systems are difficult to use for real-time processing. Our objective is to design a causal system which approximates spline interpolation. This is formulated as a problem of designing a stable inverse of a system with unstable zeros. For this purpose, we adopt H^∞ optimization. We give a closed form solution to the H^∞ optimization in the case of the cubic spline. For higher order splines, the optimal filter can be effectively solved by a numerical computation. We also show that the optimal FIR (Finite Impulse Response) filter can be designed by an LMI (Linear Matrix Inequality), which can also be effectively solved numerically. A design example is presented to illustrate the result.

Index Terms— spline functions, interpolation, H^∞ optimization

1. INTRODUCTION

Signal interpolation has many applications; it is used for curve fitting, signal reconstruction, and sampling rate conversion including resolution conversion of digital images. Many methods have been proposed for signal interpolation such as polynomial splines [4, 5] and exponential splines [6]. Polynomial splines are, in particular, widely used in image processing.

In polynomial (or exponential) spline interpolation, it is assumed that the original signal is a piece-wise polynomial (or exponential) function. Then, intersample values are computed via the Fourier coefficients relative to the spline bases. However, the coefficients are computed by using the future samples as well as the present and the past ones, and hence the interpolation system becomes non-causal. The same nature applies to the signal reconstruction by Shannon sampling theorem [3]. In the case of image processing, non-causality is not a restriction, and hence spline interpolation is widely used in that field. It is however difficult to use the splines for real-time processing such as instrumentation or audio/speech processing.

We therefore propose to design a causal system which approximates spline interpolation. This is formulated as a problem of designing a stable inverse of a system with unstable zeros. For this purpose, we adopt H^∞ optimization. By this, we can obtain the H^∞-optimal stable inverse, and hence causal spline interpolation is obtained. Moreover, by assuming that the filter to be designed is an FIR (Finite Impulse Response) filter, the optimization is reducible to an LMI (Linear Matrix Inequality), which can be effectively solved by, for example, standard MATLAB routines. In this article, we discuss polynomial spline interpolation. Exponential spline interpolation can be discussed in the same way.

2. SPLINE INTERPOLATION

Consider $x \in V^N$, where V^N is the space of polynomial splines of order N, which is defined as [4],

$$V^N = \left\{ x(\cdot) = \sum_{k=-\infty}^{\infty} c(k)\phi(\cdot - k), c \in \ell^2 \right\}.$$

In this equation, $\phi(t)$ is the symmetrical spline of order N, that is,

$$\phi(t) = (\underbrace{\beta^0 * \cdots * \beta^0}_{N+1})(t), \quad \beta^0(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

where '$*$' stands for convolution.

The sampled signal $x(n)$, $n = 0, 1, 2, \ldots$ of $x(t) \in V^N$ is given by

$$x(n) = \sum_{k=-\infty}^{\infty} c(k)\phi(n-k) = (c * \phi)(n). \qquad (1)$$

On the other hand, the fast sampled signal $x_L(n) := x(n/L)$, $n = 0, 1, 2, \ldots$ is given by [4]

$$x_L(n) = (c_L * \phi_L)(n), \qquad (2)$$

where $\phi_L(n) := \phi(n/L)$ and $c_L(n) := \{(\uparrow L)c\}(n)$. By (1) and (2), the spline interpolation system is composed of two filters ψ and ϕ_L, and the upsampler $\uparrow L$ as shown in Fig. 1, where ψ is a system such that

$$\psi * \phi = I. \qquad (3)$$

This is for the perfect reconstruction without any delay. If we allow a delay $d \ge 0$ for reconstruction, the condition becomes

$$\psi * \phi = z^{-d}. \qquad (4)$$

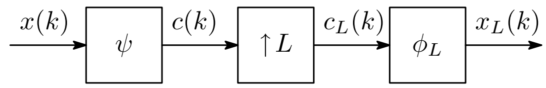

Fig. 1. Spline interpolation

3. CAUSAL SPLINE INTERPOLATION BY H^∞ OPTIMIZATION

3.1. Non-causal interpolation by decomposition

The Nth-order spline $\phi(t)$ is supported in $[0, N+1)$, and hence the sampled signal $\phi(n)$ or $\phi_L(n)$ is represented as an FIR (finite impulse response) filter. For example, in the case of $N = 3$ (cubic spline), we have

$$\phi(z) = \frac{1}{6} + \frac{2}{3}z^{-1} + \frac{1}{6}z^{-2}. \tag{5}$$

By (3), the filter $\psi(z)$ is the inverse $\psi = \phi^{-1}$ and given by $\psi(z) = 6z^2/(z^2 + 4z + 1)$. One of the poles of $\psi(z)$ lies out of the unit circle, and hence the filter $\psi(z)$ is unstable. The same thing is said of the other Nth-order splines [5]. A practical way to implement the filter is to decompose $\psi(z)$ into a cascade of stable causal and anti-causal filters [5]. In the case of the cubic spline, we first shift the impulse response of (5) as $\phi(z) = (1/6)z + (2/3) + (1/6)z^{-1}$, and then decompose $\psi(z) = \phi(z)^{-1}$ as

$$\psi(z) = -\frac{6\alpha}{1-\alpha^2}\left(\frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z} - 1\right),$$

where $\alpha = -2 + \sqrt{3}$. Since $|\alpha| < 1$, this is a stable and non-causal IIR (infinite impulse response) filter.

3.2. Causal interpolation by H^∞ optimization

In image processing, causality is not necessary, and the non-causal filter mentioned above is used widely in that field. However, it is difficult to use such non-causal filters for real-time processing, for example, in instrumentation or audio/speech processing. We therefore propose designing a causal filter $\psi(z)$ which approximates the condition (4) of delayed perfect reconstruction. Our problem is formulated as follows.

Problem 1 *Given a stable transfer function $\phi(z)$ and delay $d \geq 0$, find the causal and stable filter $\psi(z)$ which minimizes*

$$\begin{aligned} J(\psi) &= \|z^{-d} - \psi(z)\phi(z)\|_\infty \\ &= \sup_{\theta\in[0,2\pi)} |e^{-jd\theta} - \psi(e^{j\theta})\phi(e^{j\theta})|. \end{aligned} \tag{6}$$

This is a standard H^∞ optimization problem, and it can be effectively solved by standard MATLAB routines (e.g., robust control toolbox [1]) by using the design block diagram shown in Fig. 2.

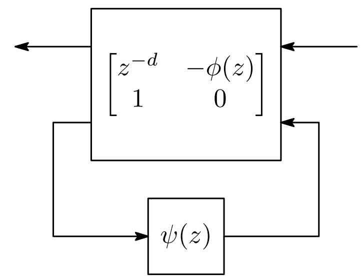

Fig. 2. H^∞ optimization

3.3. H^∞ optimal cubic spline

The cubic spline ($N = 3$) is widely used because of its simple structure. We here give the H^∞ optimal filter ψ in a closed form in the case of the cubic spline.

Define $E(z) := z^{-d} - \psi(z)\phi(z)$. Substituting (5) into this equation, we have $E(z) = z^{-d} - \psi(z)(z-\alpha_1)(z-\alpha_2)/(6z^2)$ where $\alpha_1 = -2 - \sqrt{3}$ and $\alpha_2 = -2 + \sqrt{3}$. This equation gives $\psi(z) = 6z^2(z^{-d} - E(z))/\{(z-\alpha_1)(z-\alpha_2)\}$. Since $|\alpha_1| > 1$, the filter $\psi(z)$ may have a pole which lies out of the open unit disc $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$. It is easily shown that the filter $\psi(z)$ is stable (i.e., all poles of $\psi(z)$ lie in $\mathcal{D}$) if and only if

$$E(\alpha_1) = \alpha_1^{-d}. \tag{7}$$

Then our problem is to find a stable $E(z)$ of minimum H^∞ norm under the interpolation constraint (7). This is a kind of *Nevanlinna-Pick interpolation problem* [9]. By the maximum modulus principle, we have

$$\|E\|_\infty = \sup_{|z|=1} |E(z)| = \sup_{|z|\geq 1} |E(z)| \geq |E(\alpha_1)| = |\alpha_1|^{-d}.$$

The minimum infinity norm interpolating function is therefore the constant function $E(z) = \alpha_1^{-d}$, $\|E\|_\infty = |\alpha_1|^{-d}$. By this, we obtain the optimal $\psi(z)$

$$\begin{aligned} \psi(z) &= \frac{6z^2}{(z-\alpha_1)(z-\alpha_2)}(z^{-d} - \alpha_1^{-d}) \\ &= -\frac{6z^2}{\alpha_1^d z^d (z-\alpha_2)} \sum_{k=0}^{d-1} \alpha_1^{d-1-k} z^k. \end{aligned} \tag{8}$$

Remark 1 In the case of higher order splines (i.e., $N \geq 4$), the optimal filter can be obtained by *Nevanlinna algorithm* [9]. A closed form solution is however very complicated when $N \geq 4$. In that case, the numerical computation mentioned in the previous section is available.

3.4. FIR filter design via LMI

The H^∞ optimal filter is generally an IIR one. Since the filter $\psi(z)$ to be designed is linearly dependent upon the error system $E(z) = z^{-d} - \psi(z)\phi(z)$. By this nature, the optimal

FIR filter with fixed order is obtained by optimization with a linear matrix inequality (LMI).

We here design the H^∞ optimal filter $\psi(z)$ as an FIR one

$$\psi(z) = \sum_{k=0}^{N} a_k z^{-k}.$$

A state space representation of this FIR filter is given by

$$\psi(z) = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ \hline a_N & \dots & \dots & \dots & a_1 & a_0 \end{array}\right](z)$$
$$=: \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi(\alpha) & D_\psi(\alpha) \end{array}\right](z),$$

where $\alpha := \left[\begin{array}{cccc} a_N & \dots & a_1 & a_0 \end{array}\right]$, and we use the notation by Doyle [11]:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) := C(zI - A)^{-1}B + D.$$

Note that the parameter α to be designed is linearly dependent on the matrices $C_\psi(\alpha)$ and $D_\psi(\alpha)$. Set state space representation of $\phi(z)$ and z^{-d} respectively by

$$\phi(z) =: \left[\begin{array}{c|c} A_\phi & B_\phi \\ \hline C_\phi & D_\phi \end{array}\right](z), \quad z^{-d} =: \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & 0 \end{array}\right](z).$$

Then, a state space representation of the error system $E(z) := z^{-d} - \psi(z)\phi(z)$ is given by

$$E(z) = \left[\begin{array}{ccc|c} A_\psi & B_\psi C_\phi & 0 & -B_\psi D_\phi \\ 0 & A_\phi & 0 & -B_\phi \\ 0 & 0 & A_d & B_d \\ \hline C_\psi(\alpha) & D_\psi(\alpha)C_\phi & C_d & -D_\psi(\alpha)D_\phi \end{array}\right](z)$$
$$=: \left[\begin{array}{c|c} A & B \\ \hline C(\alpha) & D(\alpha) \end{array}\right](z).$$

By this, the parameter α to be designed is linearly dependent on the matrices $C(\alpha)$ and $D(\alpha)$. By using the bounded real lemma, we can describe our design problem as an LMI [10].

Proposition 1 *Let γ be a positive number. Then the inequality $\|E(z)\|_\infty < \gamma$ holds if and only if there exist a positive definite matrix $P > 0$ such that*

$$\left[\begin{array}{ccc} A^T PA - P & A^T PB & C(\alpha)^T \\ B^T PA & -\gamma I + B^T PB & D(\alpha)^T \\ C(\alpha) & D(\alpha) & -\gamma I \end{array}\right] < 0. \quad (9)$$

Remark 2 Zeros of $E(z)$ can be set by

$$C(\alpha)(z_i I - A)^{-1}B + D(\alpha) = 0, \quad i = 1, 2, \dots, \ell.$$

This is a linear matrix equation. The LMI (9) combined with this linear constraint is also easily solved.

4. SNR PERFORMANCE ANALYSIS

In the previous section, we have proposed the H^∞ optimization design of the filter $\phi(z)$ which approximates the delayed perfect reconstruction condition (4). In this section, we analyze the overall performance of the interpolation system shown in Fig. 1. We here show that the approximation of the equation (4) is proper for increasing the SNR (signal-to-noise ratio) of the interpolation system.

Proposition 2 *Assume that $\phi(z)$ and $\psi(z)$ are causal and stable, and $x \in \ell^2$. Let $\widetilde{x}_L$ be the output of the approximated interpolation system, that is, $\widetilde{x}_L := \phi_L(\uparrow L)\psi x$. Then there exist a real number $C > 0$ which depends only on ϕ and L such that*

$$\frac{\|z^{-dL}x_L - \widetilde{x}_L\|_2}{\|x\|_2} \le C\|z^{-d} - \psi(z)\phi(z)\|_\infty. \quad (10)$$

Proof. Let ψ_I be the ideal filter which satisfies $\psi_I * \phi = z^{-d}$. Then, by (1), we have $\psi_I x = z^{-d}c$, and

$$\begin{aligned} z^{-dL}x_L - \widetilde{x}_L &= \phi_L(\uparrow L)\psi_I x - \phi_L(\uparrow L)\psi x \\ &= \phi_L(\uparrow L)z^{-d}c - \phi_L(\uparrow L)\psi\phi c \\ &= \phi_L(\uparrow L)(z^{-d} - \psi\phi)c. \end{aligned}$$

Since $\{\phi(\cdot - k)\}_{k=0}^\infty$ is a Riesz basis [2], there exists a real number $K > 0$ which depends on ϕ and is independent of c and x such that $\|c\|_2 \le K\|x\|_2$. Finally, we have

$$\begin{aligned} \|z^{-dL}x_L - \widetilde{x}_L\|_2 &\le \|\phi_L(\uparrow L)\|_\infty \|z^{-d} - \psi\phi\|_\infty \|c\|_2 \\ &\le \|\phi_L(\uparrow L)\|_\infty \|z^{-d} - \psi\phi\|_\infty K\|x\|_2 \\ &= C\|z^{-d} - \psi\phi\|_\infty \|x\|_2, \end{aligned}$$

where $C := K\|\phi_L(\uparrow L)\|_\infty$. □

By this proposition, we conclude that if the H^∞ norm of the error system $z^{-d} - \phi(z)\psi(z)$ is adequately small, the SNR of the interpolator can be increased, and hence H^∞ optimization provides a good approximation of the ideal (i.e., non-causal) spline interpolation.

5. DESIGN EXAMPLE

We here present a design example of causal spline interpolation. We consider the spline of order $N = 3$ (cubic spline). We here take the reconstruction delay $d = 3$ and design the H^∞ optimal IIR filter by (8) and FIR one with 5 taps by the linear matrix inequality (9). In the case of the cubic spline, the H^∞ optimal IIR filter (8) with $d = 3$ is given by

$$\psi(z) = \frac{-6z^2 - 6\alpha_1 z - 6\alpha_1^2}{\alpha_1^3 z(z - \alpha_2)}.$$

For comparison, we also design 5-tap FIR filter by the constrained least square design (CLSD) [7] and the Kaiser windowed approximation (KWA) [8]. Table 1 shows the coefficients of the H^∞ optimal FIR filter, the filters by CLSD and

Table 1. Coefficient a_k of FIR filter $\psi(z)$

k	H^∞ optimal	CLSD [7]	KWA [8]
0	0.1152359	0.0991561	0.06049527
1	-0.4614954	-0.4599156	-0.37739071
2	1.7307475	1.7215190	1.63379087
3	-0.4614951	-0.4599156	-0.37739071
4	0.1152352	0.0991561	0.06049527

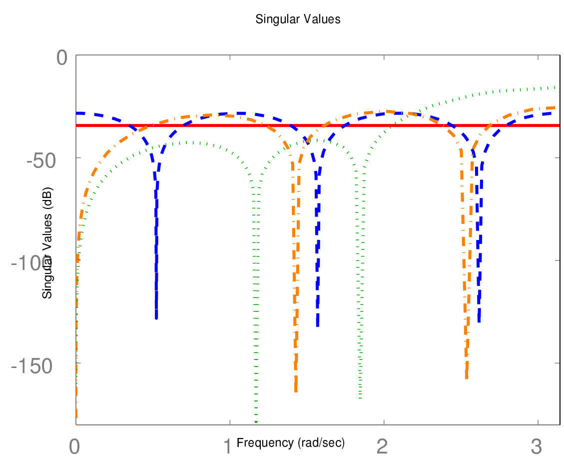

Fig. 3. Magnitude plot of $E(z)$: H^∞ optimal IIR (solid), H^∞ optimal FIR (dash), CLSD [7] (dash-dots), and KWA [8] (dots).

by KWA. Fig. 3 shows the magnitude of the frequency response of $E(z) = z^{-3} - \phi(z)\psi(z)$. By this figure, we can see that the H^∞ optimal IIR filter has an allpass characteristic. The H^∞ optimal FIR filter shows almost the same as the CLSD filter except for the zero frequency. This is because CLSD aims at exact inversion for DC signals. At the price of that, the CLSD filter shows much error in the high frequency. The KWA filter shows the same nature. Table 2 shows the H^∞ norm of the error system $E(z)$. In general, the purpose of H^∞ design is to minimize the error in the worst case. It follows that the system optimized by H^∞ design is *robust* against uncertainty of input signals. This is efficiency of H^∞ design. On the other hand, CLSD is a design to minimize the mean value of the error. Which is better depends on its application. If we do not have much information of the input, H^∞ design (worst case optimization) is more suitable.

6. CONCLUSION

In this article, a design of causal interpolation with polynomial splines has been proposed. The design is formulated as H^∞ optimization. In the case of the cubic spline, the optimal solution is given in a closed form. By using MATLAB, higher order optimal filter can be effectively solved. We have also shown that the H^∞ optimal FIR filter can be designed by an LMI. We have shown a design example to illustrate the result.

Table 2. H^∞ norm of $E(z)$

Method	$\|E\|_\infty$
H^∞ optimal IIR	0.019238
H^∞ optimal FIR	0.038597
CLSD [7]	0.053446
KWA [8]	0.16348

7. REFERENCES

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