HYBRID DESIGN OF FILTERED-X ADAPTIVE ALGORITHM VIA SAMPLED-DATA CONTROL THEORY

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ABSTRACT

Analysis and design of filtered-x adaptive algorithms are conventionally done by assuming that the transfer function in the secondary path is a discrete-time system. However, in real systems such as active noise control, the secondary path is a continuous-time system. Therefore, such a system should be analysed and designed as a hybrid system including discrete- and continuous- time systems and AD/DA devices. In this article, we propose a hybrid design taking account of continuous-time behaviour of the secondary path via lifting (continuous-time polyphase decomposition) technique in sampled-data control theory.

***Index Terms*—** Active Noise Control, Adaptive filters, Least mean square methods, Sampled data systems, Functional analysis

1. INTRODUCTION

With the advance of digital technology, it is common to use digital systems for signal processing. In particular, *active noise control*, which we study here, can adopt an advanced adaptive algorithm, by using the power of fast DSPs (digital signal processors) [1].

Fig. 1 shows a standard active noise control system. In this system, $x_c(t)$ represents a noise (or a reference signal) which enters the duct. The objective here is to eliminate the noise at the point C. To achieve this, we adopt a digital filter $K(z)$ with AD (analog-to-digital) and DA (digital-to-analog) devices. By the discretized signal x of $x_c(t)$, the filter $K(z)$ produces another digital signal y, which is converted to an analog signal by a DA converter, and then a control sound is added in the duct by a speaker B to cancel the noise.

In active noise control, it is important to compensate the distortion by the transfer characteristic of the secondary path (from B to C). To compensate this, a standard adaptive algorithm uses a filtered signal of the noise x, and is called *filtered-x algorithm* [2]. This filter is a model of the secondary path, and conventionally is a *discrete-time* one (see e.g., [2, 1]). It follows that the adaptive filter $K(z)$ optimizes the norm (or the variance in the stochastic setup) of $e(nh)$, $n = 0, 1, 2, \dots$

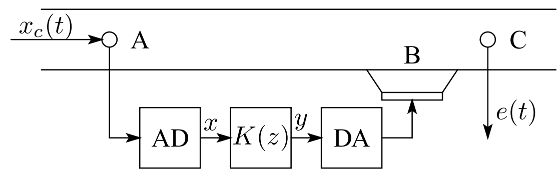

Fig. 1. Active noise control system

where h is the sampling time of AD and DA device. This is proper if the secondary path is also a discrete-time system. However, in reality, the path is a *continuous-time* system, and the optimization has to be executed taking account of the behavior of the continuous-time error signal $e(t)$. Such an optimization may seem to be difficult because the system is a *hybrid* one, which contains continuous- and discrete-time systems, AD and DA devices.

The same situation has been considered in control systems theory. The modern *sampled-data control theory* has been developed in 90's, which gives an exact design/analysis method for hybrid systems containing continuous-time plants and discrete-time controllers [3]. The key idea is *lifting*. Lifting is a transformation of continuous-time signals to a discrete-time signals. The operation can be interpreted as a *continuous-time polyphase decomposition*. In multirate signal processing, the (discrete-time) polyphase decomposition enables the designer to perform all computations at the lowest rate [4]. In the same way, by lifting, continuous-time signals or systems can be represented in the discrete-time domain with no errors (see section 3). The lifting approach is recently applied to digital signal processing [5, 6].

In line with these comtributions, this article will focus on a new design of filtered-x adaptive algorithm which takes account of the continuous-time behavior. As a filter before x, our adaptive scheme uses an analog (i.e., continuous-time) model of the secondary path. An analog model can be obtained via, for example, acoustic impedance measurement, see [7]. The proposed algorithm involves an integral computation on a finite interval. To execute this computation, we adopt an approximation based on lifting representation. The

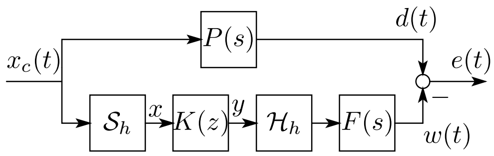

Fig. 2. Block diagram of active noise control system

approximated algorithm can be easily executed by a (linear, time-invariant, and finite dimensional) digital filter.

Throughout this paper, we use the following notations. We denote by L^2 and $L^2[0,h)$ the Lebesgue space consisting of all square integrable real functions on $[0,\infty)$ and $[0,h)$, respectively. By ℓ^2 we denote the set of all real-valued square summable sequences on $\mathbb{Z}_+ := \{0,1,2,\ldots\}$. In a Hilbert space X, the inner product is denoted by $\langle\cdot,\cdot\rangle_X$, and the norm by $\|\cdot\|_X$. For a matrix M, we denote its transpose by $M^\top$. By $(v)_i$ or $(M)_{ij}$ we denote the i-th or (i,j)-th element of a vector v or a matrix M.

2. PROBLEM FORMULATION

In this section, we formulate the design problem of active noise control shown in Fig. 1. Consider the block diagram shown in Fig. 2. In this diagram, $P(s)$ is the transfer function of the primary path from A to C in Fig. 1. The transfer function of the secondary path from B to C is represented by $F(s)$. Both $P(s)$ and $F(s)$ are continuous-time systems. We model the AD device by the ideal sampler $\mathcal{S}_h$ with a sampling period h, that is,

$$(\mathcal{S}_h x_c)[k] := x_c(kh), \quad k \in \mathbb{Z}_+.$$

The DA device is modeled by the zero-order hold $\mathcal{H}_h$ with the same period h,

$$(\mathcal{H}_h y)(t) := \sum_{k=0}^{\infty} \beta_0(t-kh)y[k], \quad t \in [0,\infty),$$

where $\beta_0(t)$ is the zero-order hold function defined by

$$\beta_0(t) := \begin{cases} 1, & t \in [0,h), \\ 0, & \text{otherwise.} \end{cases}$$

Under the setup, we consider the following optimization problem.

Problem 1 *Find the optimal FIR (finite impulse response) filter*

$$K(z) = \sum_{k=0}^{N-1} a_k z^{-k}$$

which minimizes

$$J = \int_0^\infty |e(t)|^2 \mathrm{d}t. \tag{1}$$

Instead of the conventional adaptive filter design [8], this problem deals with the continuous-time behavior of the error signal $e(t)$. To solve such a hybrid problem, the lifting approach based on the sampled-data control theory is very effective.

3. LIFTING OPERATOR

In this section, we define the lifting operator. Consider a continuous-time signal $f(t) \in L^2_{\mathrm{loc}}$. Then, the lifting operator $\mathcal{L}$ is defined by

$$(\mathcal{L}f)[k] := \underline{f}[k] := \{f(kh+\theta), 0 \le \theta < h\}, \ k \in \mathbb{Z}_+,$$

where h is a sampling period. For each k, $\underline{f}[k]$ is a function in $L^2[0,h)$, and we denote the value of $\underline{f}[k]$ at $\theta \in [0,h)$ by $\underline{f}[k](\theta)$. The lifted signal $\underline{f}$ is a series of functions in $L^2[0,h)$,

$$\underline{f} = \left\{\underline{f}[0], \underline{f}[1], \underline{f}[2], \ldots\right\}. \tag{2}$$

This is a discrete-time signal whose values are functions.

The lifting operator $\mathcal{L}$ can be interpreted as the polyphase decomposition [4] of continuous-time signals. In multirate signal processing, several sampling rates are unified in the lowest sampling rate by the polyphase decomposition. In the same way, by lifting, a continuous-time signal is represented as a discrete-time one.

The space to which the lifted signal (2) belongs is well defined due to the following lemma.

Lemma 1 *Let f be in L^2. Then,*

$$\|\mathcal{L}f\|^2 := \sum_{k=0}^{\infty} \|(\mathcal{L}f)[k]\|^2_{L^2[0,h)} < \infty.$$

By this lemma, the following ℓ^2-like space can be defined:

$$\underline{\ell}^2 := \left\{ \underline{f} : \ \sum_{k=0}^{\infty} \|\underline{f}[k]\|^2_{L^2[0,h)} < \infty \right\}.$$

The space $\underline{\ell}^2$ is a Hilbert space, and the inner product $\langle\cdot,\cdot\rangle_{\underline{\ell}^2}$ is defined by

$$\langle \underline{u}, \underline{v} \rangle_{\underline{\ell}^2} := \sum_{k=0}^{\infty} \langle \underline{u}[k], \underline{v}[k] \rangle_{L^2[0,h)}, \quad \underline{u}, \underline{v} \in \underline{\ell}^2.$$

The lifting operator has a good property shown in the following lemma [9].

Lemma 2 *For all $u, v \in L^2$, $\langle u,v\rangle_{L^2} = \langle \mathcal{L}u, \mathcal{L}v\rangle_{\underline{\ell}^2}$. Moreover, the lifting operator $\mathcal{L}$ is unitary.*

By using this property, we derive in the next section an adaptive algorithm taking account of the continuous-time behaviors of analog signals.

4. DESIGN VIA LIFTING METHOD

4.1. Hybrid design of adaptive filters

Since the lifting operator $\mathcal{L}$ preserves the inner product, the objective function (1) can be described in the lifted (i.e., discrete-time) domain:

$$J = \langle e, e\rangle_{L^2} = \langle \underline{e}, \underline{e}\rangle_{\ell^2} = \langle \underline{d}, \underline{d}\rangle_{\ell^2} - 2\langle \underline{w}, \underline{d}\rangle_{\ell^2} + \langle \underline{w}, \underline{w}\rangle_{\ell^2},$$

where $\widehat{e} := \mathcal{L}e$, $\widehat{d} := \mathcal{L}d$, and $\widehat{w} := \mathcal{L}w$. Then, by using a property that the lifted system $\underline{F\mathcal{H}_h} := \mathcal{L}F\mathcal{H}_h$ becomes a discrete-time, linear, and time-invariant system (see [9]), we have

$$\langle \underline{w}, \underline{d}\rangle_{\ell^2} = \langle \underline{F\mathcal{H}_h}y, \underline{d}\rangle_{\ell^2} = \sum_{k=0}^{N-1} a_k \langle \underline{z}^{-k}\underline{u}, \underline{d}\rangle_{\ell^2},$$

$$\langle \underline{w}, \underline{w}\rangle_{\ell^2} = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_k a_l \langle \underline{z}^{-k}\underline{u}, \underline{z}^{-l}\underline{u}\rangle_{\ell^2},$$

where $\underline{u} := \underline{F\mathcal{H}_h}x$ and $\underline{z}$ is the shift operator in ℓ^2. Next, the gradient of J with respect to a_m ($m = 0, 1, \ldots, N-1$) is given by

$$\frac{\partial J}{\partial a_m} = -2\langle \underline{z}^{-m}\underline{u}, \underline{d}\rangle_{\ell^2} + 2\sum_{l=0}^{N-1} a_l \langle \underline{z}^{-m}\underline{u}, \underline{z}^{-l}\underline{u}\rangle_{\ell^2}.$$

Setting the gradient to zero and solving for a_m, the parameter $a := [a_0, a_1, \ldots, a_{N-1}]^\top$ is optimized by $a_{\text{opt}} = T_{uu}^{-1}T_{ud}$, where

$$(T_{uu})_{ij} := \langle \underline{z}^{-i+1}\underline{u}, \underline{z}^{-j+1}\underline{u}\rangle_{\ell^2},$$
$$(T_{ud})_i := \langle \underline{z}^{-i+1}\underline{u}, \underline{d}\rangle_{\ell^2}, \quad i, j = 1, \ldots, N$$

The optimal parameter a_{opt} is given by a product of matrices T_{uu}^{-1} and T_{ud}. However, each element of the matrices are calculated by an infinite sum on $k = 0, 1, 2, \ldots$ (or an integral on the whole interval $[0, \infty)$). Such a calculation is not suitable for practical processing. Moreover, only an estimate of the secondary path $F(s)$ is available, which are denoted by $\widehat{F}(s)$. Therefore, we use instantaneous estimates for T_{uu} and T_{ud}, and adopt an LMS type of algorithm as follows:

$$a[n+1] = a[n] + \mu\widehat{T}_{ue}[n], \quad n \in \mathbb{Z}_+, \tag{3}$$

where $\widehat{\underline{u}} := \underline{\widehat{F}\mathcal{H}_h}x$ and

$$(\widehat{T}_{ue}[n])_i := \langle \widehat{\underline{u}}[n-i+1], \underline{e}[n]\rangle_{L^2[0,h)}, \quad i = 1, \ldots, N.$$

Fig. 3 shows the block diagram of this filtered-x adaptive system. We should notice that the input x is filtered by a copy $\widehat{F}\mathcal{H}_h$ of the secondary path including the hold device $\mathcal{H}_h$.

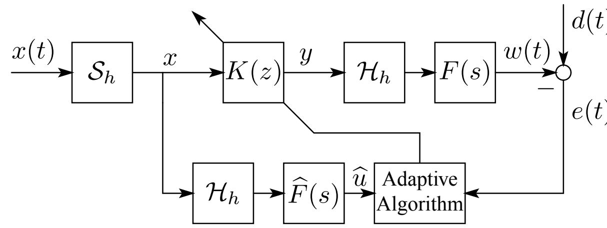

Fig. 3. filtered-x adaptive system

4.2. Approximation of adaptive algorithm

To execute the algorithm (3), we have to calculate the $L^2[0,h)$ inner product $\langle \widehat{\underline{u}}[n-m], \underline{e}[n]\rangle_{L^2[0,h)}$, $m = 0, \ldots, N-1$. The exact value of this is calculated by an integral on $[0, h)$, and is difficult to obtain in practice. Therefore, we introduce an approximation method for this computation.

First, we split the interval $[0, h)$ into L short intervals $[0, h/L)$, $[h/L, 2h/L)$, ..., $[h - h/L, h)$. Assume that the error e is constant on each short interval [1], we have

$$\langle \widehat{\underline{u}}[n-m], \underline{e}[n]\rangle_{L^2[0,h)} = e_d[n]^\top \widehat{u}_d[n-m],$$

where

$$(e_d[n])_i := \underline{e}[n]((i-1)h/L)^\top = e(nh + (i-1)h/L)^\top,$$

$$(\widehat{u}_d[n])_i := \int_{(i-1)h/L}^{ih/L} \widehat{\underline{u}}[n-m](\theta)\mathrm{d}\theta, \quad i = 1, \ldots, L.$$

Then the integral in $\widehat{u}_d[n]$ can be computed via a state-space representation of $\underline{\widehat{F}\mathcal{H}_h}$ [9]. Assume that a state-space representation of the model $\widehat{F}$ for the secondary path is given by

$$\widehat{F}: \begin{cases} \dot{x}(t) = Ax(t) + Bv(t), \\ u(t) = Cx(t), \quad t \in [0, \infty). \end{cases}$$

Then, $\widehat{u}_d[n]$ is effectively computed by the following digital filter,

$$\widehat{F}_d \begin{cases} \xi[n+1] = A_d\xi[n] + B_d x[n], \\ \widehat{u}_d[n] = C_d\xi[n] + D_d x[n], \end{cases}$$

where A_d, B_d, C_d, and D_d are matrices defined by

$$A_d := e^{Ah}, \quad B_d := \int_0^h e^{A\tau}B\mathrm{d}\tau,$$

$$(C_d)_i := \int_{(i-1)h/L}^{ih/L} Ce^{A\theta}\mathrm{d}\theta,$$

$$(D_d)_i := \int_{(i-1)h/L}^{ih/L}\int_0^\theta Ce^{A(\theta-\tau)}B\mathrm{d}\tau\mathrm{d}\theta, \quad i = 1, \ldots, L.$$

Note that the integrals in B_d, C_d, and D_d can be effectively computed by using matrix exponential, see [10, 9]. We show the proposed adaptive scheme in Fig. 4.

[1] This means that the desired signal $d(t)$ does not contain high frequency components such that $d(t)$ does not oscillate in each short interval.

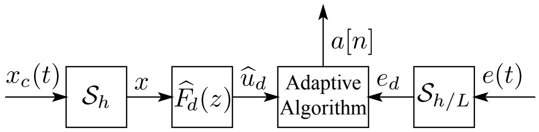

Fig. 4. filtered-x adaptive scheme

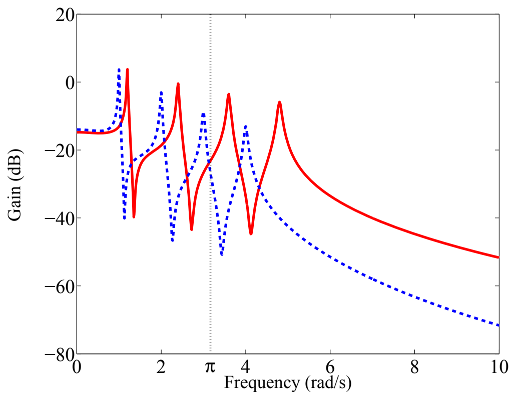

Fig. 5. Freqnecy response of $F(s)$ (solid) and $P(s)$ (dash). The dotted line shows the Nyquist frequency.

5. SIMULATION

In this section, we simulate our adaptive filtering. Set the sampling period $h = 1$. The fast-sampling ratio L is 10. The transfer functions $F(s)$ and $P(s)$ have peaks at $\omega = 1, 2, 3, 4$ and $\omega = 1.2, 2.4, 3.6, 4.8$, respectively. These frequency responses are shown in Fig. 5. Note that these systems have peaks beyond the Nyquist frequency $\omega = \pi$. Let the input $x_c(t)$ be the white Gaussian signal with zero-mean and variance 1, and Fig. 6 shows the square error $|e(t)|^2$. We compare our result with a conventional design (i.e., a discrete-time design [1]). Our result shows better response than the conventional one, since our method takes account of the continuous-time characteristic, in particular, high frequency components beyond the Nyquist frequency.

6. CONCLUSION

In this article, we have proposed a hybrid design of filtered-x adaptive algorithm via lifting method in sampled-data control theory. The proposed algorithm can take account of the continuous-time behavior of the error signal. We have also proposed an approximation of the algorithm, which can be easily implemented in DSP. A simulation result shows the effectiveness of our method.

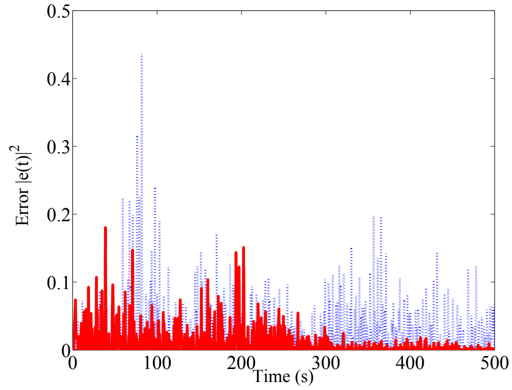

Fig. 6. Square error $|e(t)|^2$; proposed design (solid) and discrete-time design (dotted)

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