OPTIMAL DESIGN OF FRACTIONAL DELAY FIR FILTERS WITHOUT BAND-LIMITING ASSUMPTION

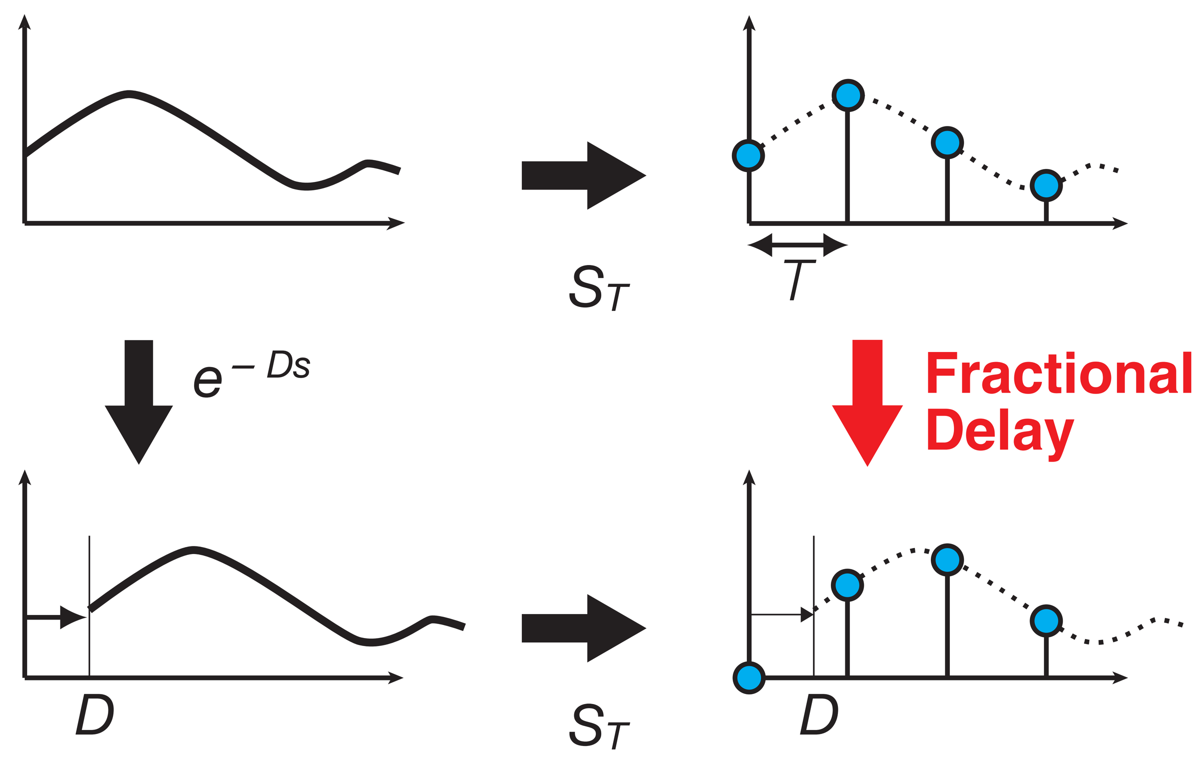
Masaaki Nagahara and *Yutaka Yamamoto*

Department of Applied Analysis and Complex Dynamical Systems,
Graduate School of Informatics, Kyoto University,
Kyoto 606-8501, Japan

Abstract

Fractional delay filters are those that are designed to delay the input signals by a fractional amount of the sampling time. Since the delay is fractional, the **intersample behavior of original analog signals become crucial**. While the conventional design bases itself on the assumption that the incoming analog signals are fully band-limited up to the Nyquist frequency, the present paper applies the modern **sampled-data H^∞ optimization** which aims at **restoring the intersample behavior without the band-limiting assumption**. It is shown that the optimal FIR filter design is reducible to a convex optimization described by a **linear matrix inequality** (**LMI**). A design example is shown to illustrate the advantage of the proposed method.

How Do Fractional Delay Filters Work?

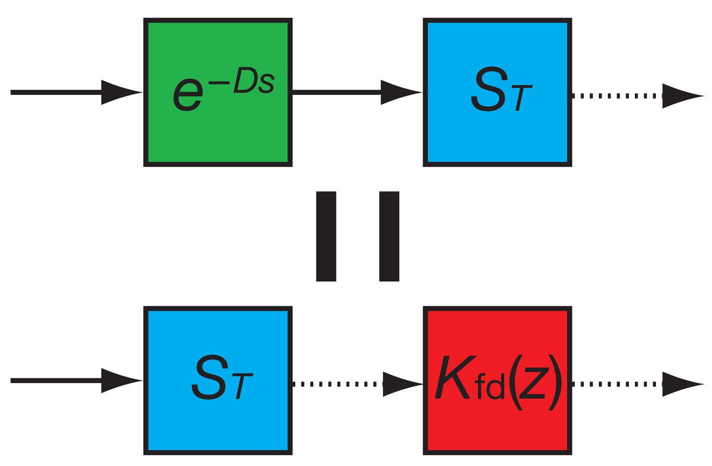


D is not an integer multiple of sampling time *T*.

Conventional Design

If the input analog signals are **fully band-limited** up to the **Nyquist frequency**, we have the equation:

$$S_T e^{-Ds} = K_{\mathrm{fd}} S_T$$



The solution (the **ideal filter**) is given by

$K_{\mathrm{fd}}(e^{j\omega T}) = e^{-j\omega DT}$ (frequency response)

or

$k_{fd}[n] = \mathrm{sinc}(nT - D), \quad \mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ (impulse response)

The ideal filter is **non-causal** and **infinite-dimensional**.

Conventional design aims at **approximating the ideal filter** via

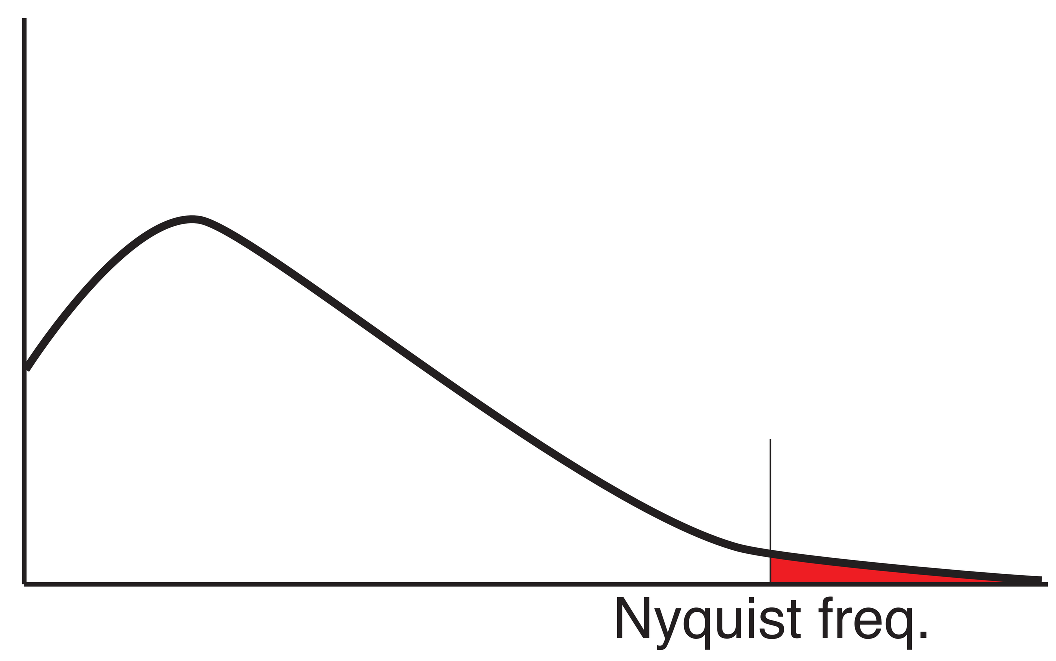
(1) Window method (with impulse response)

(2) Maximally-flat FIR approximation (with frequency response)

(3) Weighted least-squares approximation (with frequency response)

However

Real analog signals are **never fully band-limited**. They contain some frequency components **beyond the Nyquist frequency**.



We Propose

• Design with the frequency characteristic of the input **analog signals** which are not fully band-limited up to the Nyquist Frequency.

• Optimization with the **H^∞ norm** of the error system which both analog and digital signals.

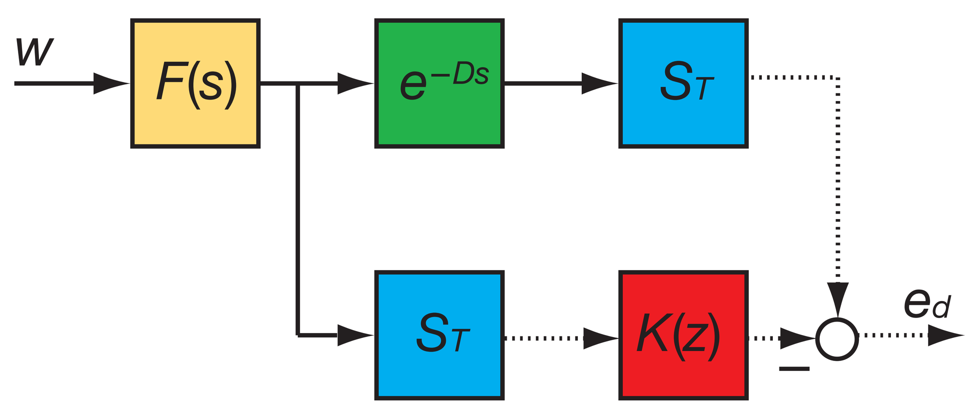
• Formulated as a **sampled-data H^∞ optimization** Problem.

Design Problem

Given $F(s)$, $D>0$, and $T>0$, find the digital filter $K(z)$ minimizing

$$\|\mathcal{E}\|_\infty := \sup_{w \in L^2} \frac{\|\mathcal{E}w\|_{\ell^2}}{\|w\|_{L^2}}$$

$$\mathcal{E} := (S_T e^{-Ds} - K(z) S_T) F(s)$$



The analog filter $F(s)$ governs the **frequency-domain characteristic** of the input signal w. $F(s)$ is conventionally assumed to be an **ideal lowpass** filter whose cut-off freq. is the Nyquist freq. Here, **we do not assume such full band-limiting**.

The error system $\mathcal{E}$ has both continuous- and discrete-time signals. We discretize the continuous-time signals **preserving H^∞ norm** of the error system, and find the optimizing filter in the discrete-time domain. This is a **sampled-data H^∞ optimization** problem.

Reduction to a Finite Dimensional Linear Time Invariant System

The error system $\mathscr{E}$ is an **infinite-dimensional** system since

(1) $\mathscr{E}$ is an operator from L^2 to ℓ^2,

(2) $\mathscr{E}$ contains a time delay.

By using **sampled-data control theory**, we can obtain a **finite-dimensional** (FD) **linear time-invariant** (LTI) system whose H^∞ norm is equal to that of the system $\mathscr{E}$.

Theorem 1

For the error system $\mathscr{E} \in B(L^2, \ell^2)$, there exists an FD LTI system E_d such that

$$\|\mathscr{E}\|_\infty = \|E_d\|_\infty$$

Outline of Proof

(1) Dual operator $\mathscr{E}^*$ of $\mathscr{E}$ such that $(\mathscr{E}w, v)_{\ell^2} = (w, \mathscr{E}^* v)_{L^2}$

(2) $\|\mathscr{E}\|_\infty^2 = \|\mathscr{E}\mathscr{E}^*\|$

(3) $\mathscr{E}\mathscr{E}^* \in B(\ell^2, \ell^2)$, and there exists a finite-dimensional discrete-time system $E_d \in B(\ell^2, \ell^2)$ such that $E_d E_d^* = \mathscr{E}\mathscr{E}^*$.

(4) $\|\mathscr{E}\|_\infty^2 = \|\mathscr{E}\mathscr{E}^*\| = \|E_d E_d^*\| = \|E_d\|_\infty^2$ ❑

Discrete-time H^∞ Optimization

E_d is given as a transfer function by

$$E_d(z) = (C_1 - K(z)C_2)(zI - A_d)^{-1}B_d \qquad (🍎)$$

where C_1, C_2, A_d and B_d are matrices.

Assume $K(z)$ is an **FIR filter**,

$$K(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}$$

and we can rewrite (🍎) as

$$E_d(z) = C(\alpha)(zI - A)^{-1}B$$

where $C(\alpha)$, A, B are matrices. Note that $C(\alpha)$ is **linearly dependent on** $\alpha = [a_0, a_1, \dots, a_N]$, and A and B are **independent of** α.

Our problem is then reduced as follows:

Reduced Problem

Find the FIR parameter α minimizing

$$\|E_d\|_\infty := \sup_{\theta \in [0, \pi)} |C(\alpha)(e^{j\theta}I - A)^{-1}B|$$

Theorem 2

Assume $\gamma > 0$. Then $\|E_d\|_\infty < \gamma$ if and only if there exists a matrix $P > 0$ such that

$$\begin{bmatrix} A^TPA - P & A^TPB & C(\alpha)^T \\ B^TPA & -\gamma I + B^TPB & 0 \\ C(\alpha) & 0 & -\gamma I \end{bmatrix} < 0 \qquad (🍎🍎)$$

Outline of Proof

(1) By bounded real lemma (Kalman-Yakubovich-Popov lemma), $\|E_d\|_\infty < \gamma$ is equivalent to that $\exists\, P > 0$ such that

$$\begin{bmatrix} A & B \\ C(\alpha) & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C(\alpha) & 0 \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix}$$

(2) By Schur complement, this is equivalent to (🍎🍎). ❑

Design Procedure

(1) Give a delay time D, **sampling time** T, **analog filter** $F(s)$.

(2) Set FIR order N **and compute matrices** A, B **and** $C(\alpha)$.

(3) Find α **minimizing** γ **s.t.** $P > 0$, $\gamma > 0$ **and the LMI** (🍎🍎).

The optimal α in the procedure (3) can be easily obtained by standard linear optimization softwares (e.g., LMI toolbox in MATLAB). Moreover, **if $F(s)$ is a first order lowpass filter**, we can obtain the optimal filter **analytically**.

Design Example

(1) Design parameters: $D = 10.8$ [sec], $T = 1$ [sec] and $F(s)$ is

$$F(s) = \left(\frac{\omega_c}{s + \omega_c}\right)^L, \quad \omega_c = 0.5 \text{ [rad/sec]}, \quad L = 1, 2, 4, 8.$$

(2) Filter length is 32 taps (i.e., $N = 31$).

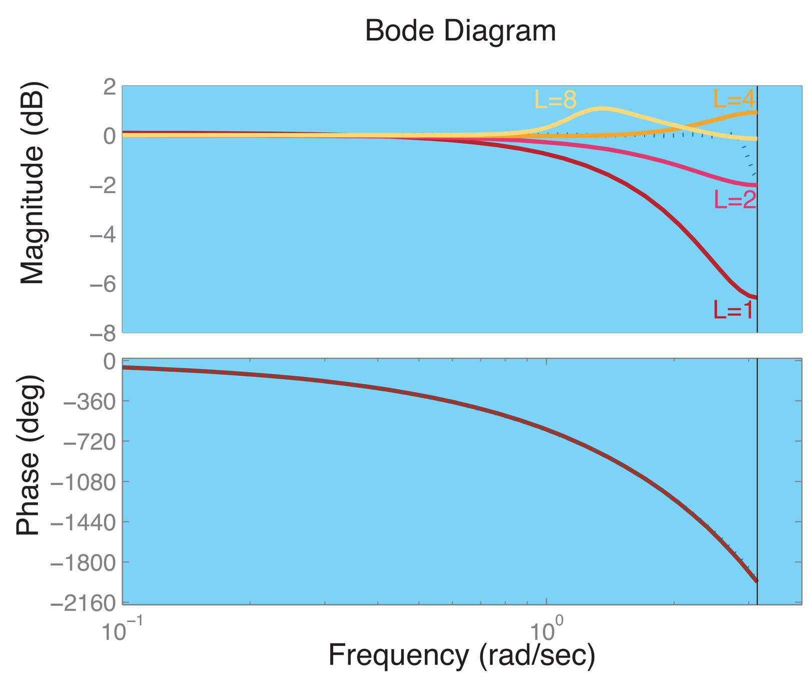


Fig. 1. Fractional delay FIR filters: sample-data H^∞ design (solid) and 32-tap FIR filter by the Kaiser window method (dots)

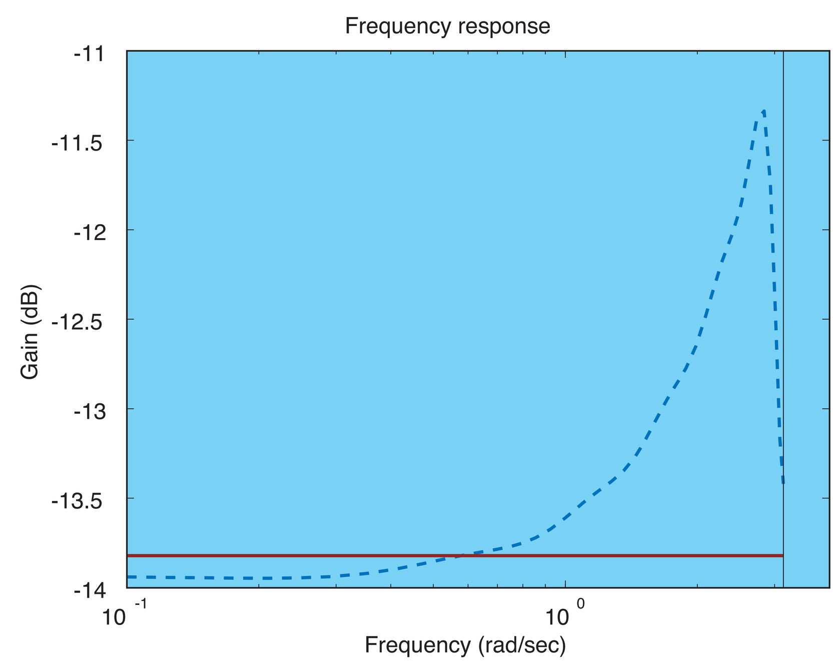


Fig. 2. Frequency response of $\mathscr{E}$: sample-data H^∞ design (solid) and 32-tap FIR filter by the Kaiser window method (dots)

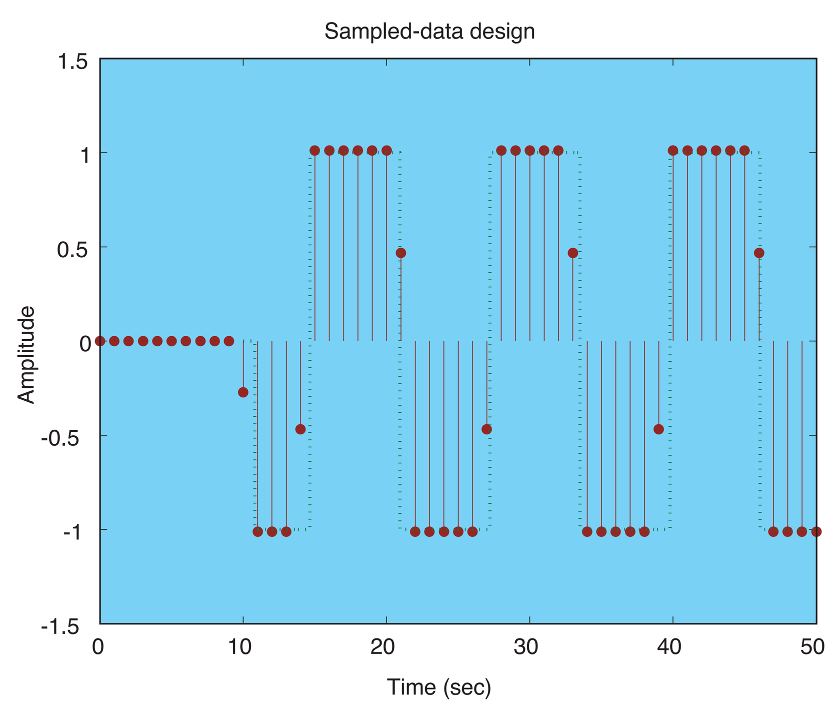


Fig. 3. Time response against a rectangular wave: sample-data H^∞ design

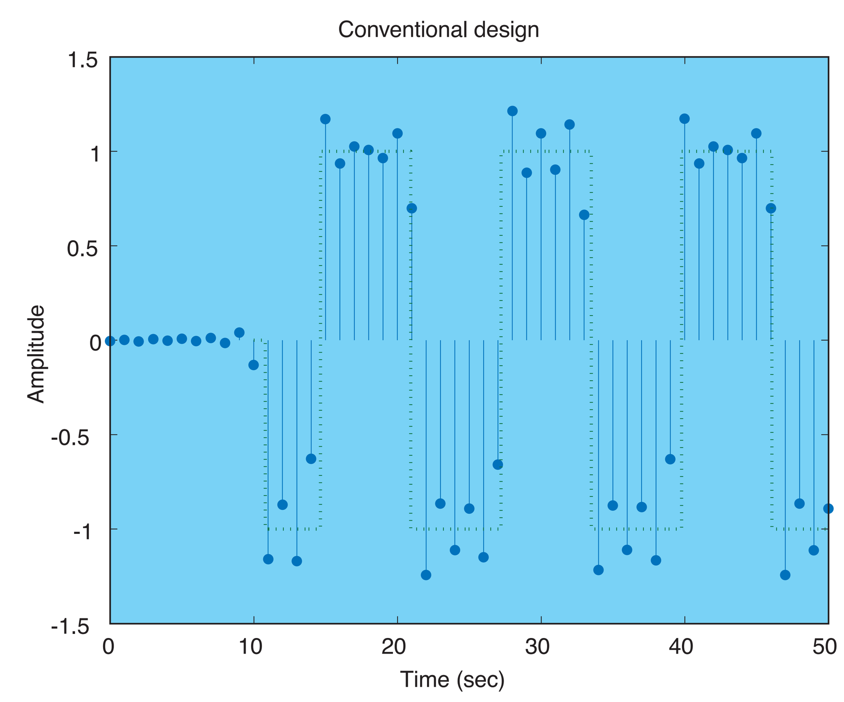


Fig. 4. Time response against a rectangular wave: 32-tap FIR filter by the Kaiser window method (dots)

Conclusion

We have presented a new method of designing fractional delay FIR filters via sampled-data H^∞ optimization. An advantage here is that an analog optimal performance can be obtained. The design problem can be reduced to a convex optimization with an LMI, which leads to an easy computation of design. The designed filter exhibits a much more satisfactory performance than conventional ones.