OPTIMAL DESIGN OF FRACTIONAL DELAY FIR FILTERS WITHOUT BAND-LIMITING ASSUMPTION

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ABSTRACT

Fractional delay filters are those that are designed to delay the input signals by a fractional amount of the sampling time. Since the delay is fractional, the intersample behavior of original analog signals become crucial. While the conventional design bases itself on the assumption that the incoming analog signals are fully band-limited up to the Nyquist frequency, the present paper applies the modern sampled-data H^∞ optimization which aims at restoring the intersample behavior without the band-limiting assumption. It is shown that the optimal FIR filter design is reducible to a convex optimization described by a linear matrix inequality (LMI). A design example is shown to illustrate the advantage of the proposed method.

1. INTRODUCTION

Fractional delay filters are to delay the input signal by a fraction of the sampling period. Such a filter has wide applications in signal processing, including digital communications, speech processing and digital modeling of musical instruments [1].

Conventionally, fractional delay filters are designed in the discrete-time domain by assuming that the incoming analog signals are fully band-limited up to the Nyquist frequency. However, this assumption is not realistic because no real analog signals are fully band-limited. Moreover, by their very nature, such filters should reconstruct intersample signal values. For such a problem, sampled-data control theory [2] provides an optimal platform. This theory is already confirmed to be effective in some digital filter design problems (e.g., [3]), and it is ideal in dealing with the continuous-time behavior.

We thus formulate the design problem of fractional delay FIR filters as a sampled-data H^∞ optimization problem. That is, we design an FIR filter which minimizes the H^∞-norm of the error system including both continuous- and discrete-time signals. We show that this design problem is reducible to a convex optimization with a linear matrix inequality (LMI). A design example is shown to illustrate the advantage of the proposed method.

Throughout this paper, we denote by L^2 the Lebesgue space consisting of all square integrable real functions on $[0,\infty)$, and by ℓ^2 the set of all real-valued square summable sequences on $\mathbb{Z}_+ := \{0,1,2,\ldots\}$. For linear spaces X and Y, we denote by $\mathcal{B}(X,Y)$ the collection of all bounded linear operators of X into Y. A^T denotes the transpose of a matrix A.

2. DESIGN PROBLEM

2.1. Fractional delay filters

Consider a continuous-time signal $v(t)$ as shown in Fig. 1 (a). The signal $v(t)$ is delayed by the continuous-time delay

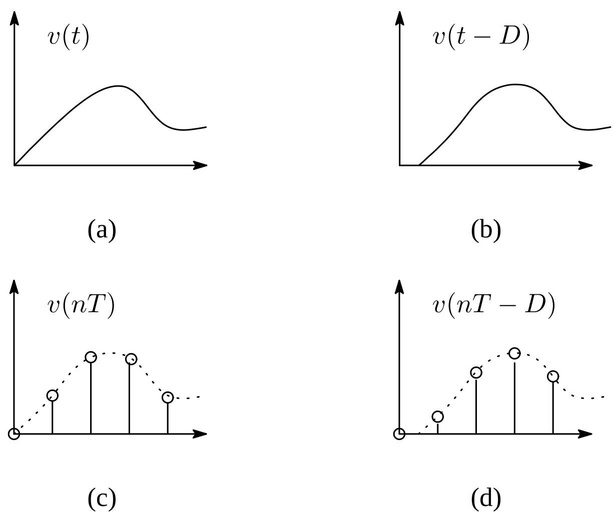


Fig. 1. (a) continuous-time signal $v(t)$, (b) delayed signal $v(t-D)$, (c) sampled signal $v(nT)$, (d) delayed and sampled signal $v(nT-D)$

operator e^{-Ds} ($D > 0$) as shown in Fig. 1 (b). Then the delayed signal $v(t-D)$ is sampled with period T and becomes a discrete-time signal $v(nT-D)$, $n \in \mathbb{Z}_+$ as shown in Fig. 1 (d).

On the other hand, consider the sampled signal $v(nT)$, $n \in \mathbb{Z}_+$ as shown in Fig. 1 (c). Then we define the ideal fractional delay filter as follows:

Definition 1 The *ideal fractional delay filter* K^{id} with delay time D is defined by

$$K^{\mathrm{id}} : v(nT) \mapsto v(nT-D).$$

Note that if $D = kT$, $k \in \mathbb{Z}_+$, the ideal filter K^{id} is the discrete-time delay z^{-k}. Moreover, if the input analog signal $v(t)$ is fully band-limited up to the Nyquist frequency $\omega_N := \pi/T$, the impulse response of the ideal fractional delay filter is obtained as follows [1]:

$$k^{\mathrm{id}}[n] = \frac{\sin \pi(nT-D)/T}{\pi(nT-D)/T} = \mathrm{sinc}\, \pi(nT-D)/T, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

The frequency response of this ideal filter is derived by the Fourier transform:

$$K^{\mathrm{id}}(e^{j\omega T}) = e^{-j\omega T D}, \quad \omega \leq \omega_N. \tag{2}$$

The ideal filter (1) or (2) cannot be realized since the filter is generally non-causal and infinite dimensional, and hence a conventional design aims at approximating (1) or (2) via a window method, maximally-flat FIR approximations, weighted least-squares approximation, and so forth [1, 4].

These methods are based upon the band-limiting assumption. In practice, however, analog signals always contain some frequency components beyond the Nyquist frequency as mentioned above. In what follows, we thus formulate the design problem of fractional delay filters without this assumption.

2.2. Design problem of fractional delay filters

Consider the block diagram Fig. 2. In this diagram, $F(s)$ governs the frequency-domain characteristic[1] of the input signal $w \in L^2$. The upper path of the diagram is the ideal process of the fractional delay filter (the process (a) → (b) → (d) in Fig. 1), that is, the continuous-time signal v is delayed by the continuous-time delay e^{-Ds}, sampled with period T, and becomes a discrete-time signal $z_d \in \ell^2$. On the other hand, the lower path is the real process ((a) → (c) → (d) in Fig. 1), that is, the continuous-time signal v is sampled with period T, filtered by $K(z)$ to be designed, and becomes a discrete-time signal $u_d \in \ell^2$.

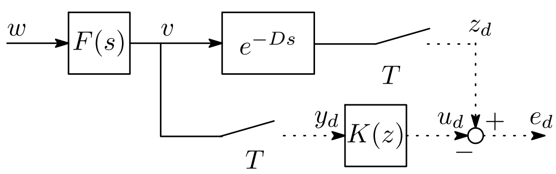


Fig. 2. Error system for designing fractional delay filter $K(z)$

Put $e_d := z_d - u_d$ (the difference between the ideal output z_d and the real output u_d), and let $\mathcal{E}$ denote the system from $w \in L^2$ to $e_d \in \ell^2$. Then our problem is to find the filter $K_D(z)$ which minimizes the H^∞ norm of the error system $\mathcal{E}$.

Problem 1 *Given a stable, strictly proper $F(s)$, delay time $D > 0$, sampling period $T > 0$, find the digital filter $K(z)$ which minimizes*

$$\|\mathcal{E}\|_\infty := \sup_{w \in L^2} \frac{\|e_d\|_{\ell^2}}{\|w\|_{L^2}}. \tag{3}$$

3. DESIGN OF FRACTIONAL DELAY FIR FILTERS

3.1. Reduction to a finite-dimensional problem

A difficulty in Problem 1 is that the error system $\mathcal{E}$ contains both continuous- and discrete-time dynamics, that is, $\mathcal{E}$ is an infinite-dimensional system. To solve this problem, we employ the modern sampled-data control theory [2]. Via this theory, we have the following theorem.

Theorem 1 *For the error system $\mathcal{E} \in \mathcal{B}(L^2, \ell^2)$, there exists a finite-dimensional discrete-time system E_d such that*

$$\|\mathcal{E}\|_\infty = \|E_d\|_\infty.$$

Outline of Proof Introduce the dual operator [5] $\mathcal{E}^*$ of $\mathcal{E}$ such that $(\mathcal{E}w, v)_{\ell^2} = (w, \mathcal{E}^* v)_{L^2}$, $w \in L^2$, $v \in \ell^2$. Since $\mathcal{E}$ is bounded, we have $\|\mathcal{E}\|_\infty^2 = \|\mathcal{E}\mathcal{E}^*\|$. Then, the operator $\mathcal{E}\mathcal{E}^*$ is in $\mathcal{B}(\ell^2, \ell^2)$, and there exists a finite-dimensional discrete-time system $E_d \in \mathcal{B}(\ell^2, \ell^2)$ such that $E_d E_d^* = \mathcal{E}\mathcal{E}^*$, and finally we have $\|\mathcal{E}\|_\infty^2 = \|\mathcal{E}\mathcal{E}^*\| = \|E_d\|_\infty^2$. For the precise discussion, see [6]. $\square$

This theorem leads to a finite-dimensional optimization problem. Our objective is then to find an FIR filter minimizing $\|E_d\|_\infty$.

[1] Conventionally, $F(s)$ is assumed to be an ideal filter such that $F(j\omega) = 0$, $|\omega| \geq \omega_N$.

3.2. Design of fractional delay FIR filter

FIR filters are often preferred to IIR filters because of their advantages in respect of implementation. In this section, we will show the design of the optimal FIR filters according to Theorem 1.

For fixed $N \geq 0$, put

$$K(z) = \sum_{n=0}^{N} a_n z^{-n},$$

and introduce a state-space realization

$$K(z) = C_K(\alpha)(zI - A_K)^{-1}B_K + D_K(\alpha), \qquad (4)$$

where $\alpha = \begin{bmatrix} a_0 & a_1 & \ldots & a_N \end{bmatrix}$,

$$A_K = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C_K(\alpha) = \begin{bmatrix} a_N & a_{N-1} & \ldots & a_1 \end{bmatrix}, \quad D_K(\alpha) = a_0.$$

Then the discrete-time system $E_d(z)$ is given as follows [6]:

$$E_d(z) = (C_1 - K(z)C_2)(zI - A_d)^{-1}B_d, \qquad (5)$$

where C_1, C_2, A_d, B_d are matrices. Substitution of (4) in (5) yields $E_d(z) = C(\alpha)(zI - A)^{-1}B$, where

$$A = \begin{bmatrix} A_d & 0 \\ B_K C_2 & A_K \end{bmatrix}, \quad B = \begin{bmatrix} B_d \\ 0 \end{bmatrix},$$

$$C(\alpha) = \begin{bmatrix} C_1 - D_K(\alpha)C_2 & -C_K(\alpha) \end{bmatrix}.$$

Note that the transfer function $E_d(z)$ is linear in α. It follows that the design problem of choosing α to minimize $\|E_d\|_\infty$ can be expected to become an LMI. In fact, the bounded real lemma [7] readily yields the following.

Theorem 2 *Assume $\gamma > 0$. Then $\|E_d\|_\infty < \gamma$ if and only if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} A^T PA - P & A^T PB & C(\alpha)^T \\ B^T PA & -\gamma I + B^T PB & 0 \\ C(\alpha) & 0 & -\gamma I \end{bmatrix} < 0. \qquad (6)$$

Proof By the bounded real lemma [7], $\|E_d\|_\infty < \gamma$ is equivalent to the condition that there exists a matrix $P > 0$ such that

$$\begin{bmatrix} A & B \\ C(\alpha) & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C(\alpha) & 0 \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Then, this inequality is equivalently converted to (6) by using the Schur complement [7]. □

Theorem 2 gives an LMI characterization for the existence of an FIR filter $K(z)$ such that $\|E_d\|_\infty < \gamma$. Whether (6) is satisfied can easily be checked, for example by standard MATLAB (particularly, LMI toolbox) routines [8]. To obtain the optimal α, minimize γ subject to the LMI (6), which can also easily be accomplished by the software.

4. DESIGN EXAMPLE

We present a design example of fractional delay filters. The design parameters are as follows: the sampling period $T = 1$ [sec], the delay $D = 10.8$ [sec], and

$$F(s) = \left(\frac{\omega_c}{s + \omega_c}\right)^L, \quad \omega_c = 0.5, \quad L = 1, 2, 4, 8.$$

Note that $F(s)$ has the cutoff frequency at 0.5 [rad/sec]. The order of our FIR filter $N = 31$. If $L = 1$, we have the analytic solution [6]

$$K(z) = z^{-m}(a_0 + a_1 z^{-1}),$$

where $D = d + mT$, $0 < d < T$, $m \in \mathbb{Z}_+$ and

$$a_0 = \frac{\sinh \omega_c(T - d)}{\sqrt{\omega_c} \sinh \omega_c T}, \quad a_1 = e^{-\omega_c T}(e^{\omega_c d} - a_0),$$

which minimizes $\|E_d\|_\infty$. Thus, we use this formula when $L = 1$ and design by Theorem 2 when $L \geq 2$. We compare these filters with a conventional filter of 31-st order FIR filter by the Kaiser window method [1].

Fig. 3 shows the Bode plots of the filters obtained by the proposed method and the conventional one. The phase responses are almost similar, whereas there is a difference in the magnitude. We can see that as L becomes larger, the frequency response approaches the ideal response (2), and the conventional filter appears best in the context of the conventional design methodology.

To see the difference, we show the frequency responses of the error system $\mathcal{E}$ (see [9]) of the proposed filter ($L = 1$) and the conventional one in Fig. 4. Our filter exhibits much smaller errors in the high-frequency domain. This is because the conventional design does not take into account the frequency response of the source analog signals while the present method does.

Then, we show the time response against rectangular waves in Fig. 5 ((a) is the proposed design and (b) is the conventional one). The present method is clearly superior to the conventional one which shows much ringing at the edges of the waves.

5. CONCLUSION

We have presented a new method of designing fractional delay FIR filters via sampled-data H^∞ optimization. An advantage here is that an analog optimal performance can be

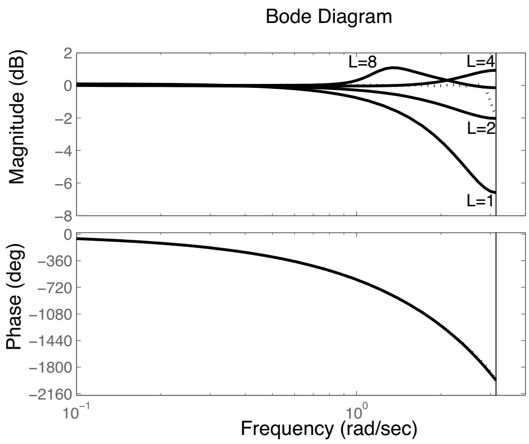


Fig. 3. Fractional delay FIR filters: H^∞ design (solid) and conventional design (dots)

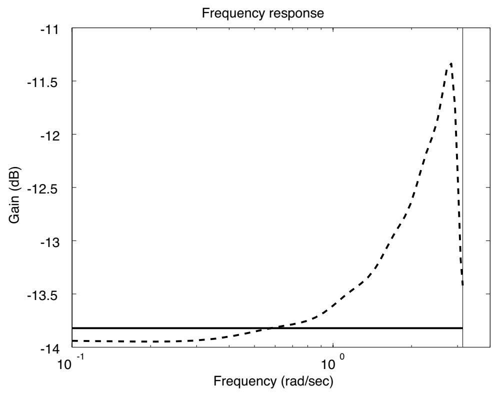
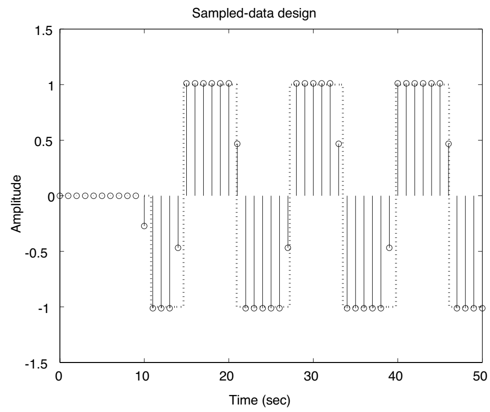
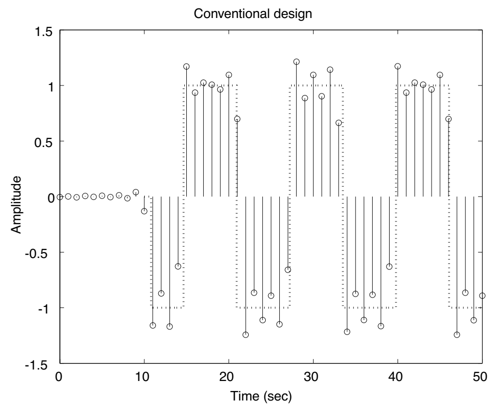


Fig. 4. Frequency response of $\mathcal{E}$: H^∞ design (solid) and conventional design (dash)



(a) H^∞ design



(b) Conventional design

Fig. 5. Time responses

obtained. The design problem can be reduced to a convex optimization with an LMI, which leads to an easy computation of design. The designed filter exhibits a much more satisfactory performance than conventional ones.

6. REFERENCES

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