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Optimal Design of Fractional Delay Filters

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***Abstract*— Fractional delay filters are those that are designed to delay the input signals by a fractional amount of the sampling time. They are widely used in digital communications, speech processing, digital modeling of musical instruments, etc. Since the delay is fractional, the intersample behavior of input signals become crucial. While the conventional design bases itself on the full band-limiting assumption, the present paper applies the modern sampled-data H^∞ control which aims at restoring the intersample behavior. It is shown that the optimal design problem is reducible to a norm-equivalent discrete-time problem. A numerical example is shown to illustrate the advantage of the proposed method.**

I. INTRODUCTION

Fractional delay filters are to delay the input signal by a fraction of the sampling time. Such a filter has wide applications in signal processing, including digital communications, speech processing and digital modeling of musical instruments [2], [6]. For example, in digital communications, the received continuous-time pulse sequences should be sampled at the middle of each pulse to decide the received bit or symbol value. For this purpose, a fractional delay filter is used to synchronize the sampler with the incoming signal [2], [6].

Conventionally, fractional delay filters are designed in the discrete-time domain by assuming that the incoming continuous-time signals are fully band-limited up to the Nyquist frequency. However, this assumption is not realistic because no real analog signals are fully band-limited. Moreover, by their very nature, such filters should reconstruct intersample signal values, and sampled-data control theory provides an optimal platform for such a problem. This theory is already confirmed to be effective in some digital filter design problems [9], [4], [5], and it is ideal in dealing with the continuous-time behavior.

We thus formulate the design problem of fractional delay filters as a sampled-data H^∞ control problem. That is, we design a filter which minimizes the H^∞-norm of the error system consisting of a continuous-time delay and a filter with ideal sampler. We show that this design problem is reducible to a discrete-time H^∞ one by lifting [8]. We also provide an analytical solution to the case in which such fractional delays are time-varying, under the assumption that the underlying frequency characteristic of the continuous-time input signal is governed by a low-pass filter of first order. While this may appear somewhat restrictive, it covers many typical cases and variations from it can be taken care of by some robustness properties. A numerical example is shown to illustrate the advantage of the proposed method.

Throughout this paper, we use the following notations. $L^2[0,\infty)$ and $L^2[0,h)$ denote the Lebesgue spaces consisting of all square integrable real functions on $[0,\infty)$ and $[0,h)$, respectively. $L^2[0,\infty)$ may be abbreviated to L^2. l^2 denotes the set of all real-valued square summable sequences, and $l^2_{L^2[0,h)}$ the set of all square summable sequences whose values are in $L^2[0,h)$. $\Re^N$ denotes the N-dimensional vector space over $\Re$. The transfer function whose realization is $\{A,B,C,D\}$ is denoted as $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. A^T denotes the transpose of a matrix A.

II. DESIGN PROBLEM OF FRACTIONAL DELAY FILTERS

A. Fractional delay filters

Consider the continuous-time signal $x(t)$ shown in Fig. 1 (a). The continuous-time signal $x(t)$ is delayed by the continuous-time delay operator e^{-Ds} ($D>0$) as shown in Fig. 1 (b). Then the delayed signal $x(t-D)$ is sampled with sampling period h and becomes a discrete-time signal $x_D[n] := x(nh-D)$, $n=0,1,2,\ldots$ (shown in Fig. 1 (d)).

On the other hand, consider the sampled signal $x[n] := x(nh)$, $n=0,1,2,\ldots$, with sampling period h as shown in Fig. 1 (c). Then we define the ideal fractional delay filter as follows:

Definition 1: The *ideal fractional delay filter* K_D^{id} with delay time D is defined by

$$K_D^{\mathrm{id}} : x[n] = x(nh) \mapsto x_D[n] = x(nh-D).$$

Note that if $D=kh$, $k=0,1,2,\ldots$, the ideal fractional delay filter K_D^{id} is the discrete-time delay z^{-k}. Moreover, if the input analog signal $x(t)$ is fully band-limited up to the Nyquist frequency $\omega_N := \pi/h$, the impulse response of the ideal fractional delay filter is obtained as follows [2], [6]:

$$k_D^{\mathrm{id}}[n] = \frac{\sin \pi(n-D)}{\pi(n-D)} = \mathrm{sinc}(n-D), \quad n = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

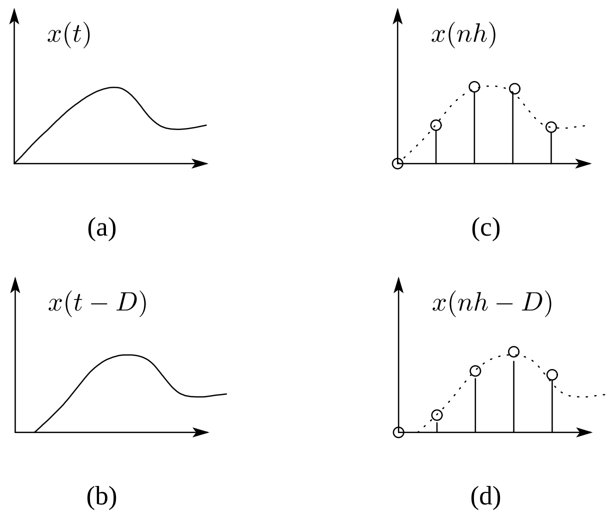

Fig. 1. (a) continuous-time signal $x(t)$, (b) delayed signal $x(t-D)$, (c) sampled signal $x(nh)$, (d) delayed and sampled signal $x(nh-D)$

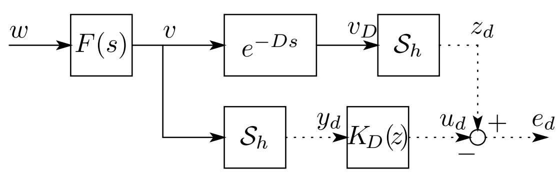

Fig. 2. Error system for designing fractional delay filter K_D

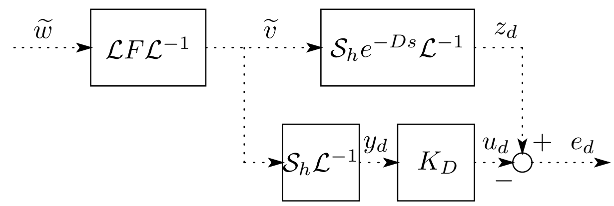

Fig. 3. Lifted error system

The frequency response of this ideal filter is derived by the Fourier transform:

$$K_D^{\mathrm{id}}(e^{j\omega h}) = e^{-j\omega D h}, \quad \omega \leq \omega_N. \tag{2}$$

The ideal filter (1) or (2) cannot be realized since the filter is generally non-causal and infinite dimensional, and hence a conventional design aims at approximating (1) or (2) via a window method, maximally-flat FIR approximations, weighted least-squares approximation, and so forth [2], [6].

These methods are based upon the assumption that the input analog signals are fully band-limited. In practice, however, analog signals always contain some frequency components beyond the Nyquist frequency. In what follows, we thus formulate the design problem of fractional delay filters by using the sampled-data H^∞ optimization.

B. Design problem of fractional delay filters

Consider the block diagram Fig. 2. In this diagram, $F(s)$ governs the frequency-domain characteristic of the input signal $w \in L^2$. The upper path of the diagram is the ideal process of the fractional filter (the process (a) → (b) → (d) in Fig. 1), that is, the continuous-time signal v is delayed by the continuous-time delay e^{-Ds}, sampled by the sampler $\mathcal{S}_h$ and becomes a discrete-time signal $z_d \in \ell^2$. On the other hand, the lower path is the real process, that is, the continuous-time signal v is sampled by the sampler $\mathcal{S}_h$, filtered by K_D (to be designed), and becomes a discrete-time signal $u_d \in \ell^2$.

Put $e_d := z_d - u_d$ (the difference between the ideal output z_d and the real output u_d), and let $\mathcal{T}_{ew}$ denote the system from $w \in L^2$ to $e_d \in \ell^2$. Then our problem is to find the filter $K_D(z)$ which minimizes the H^∞ norm of the error system $\mathcal{T}_{ew}$.

Problem 1: Given a stable, strictly proper $F(s)$, delay time $D > 0$, sampling time $h > 0$ and an attenuation level $\gamma > 0$, find a digital filter $K_D(z)$ such that

$$\|\mathcal{T}_{ew}\|_\infty := \sup_{w \in L^2} \frac{\|e_d\|_{\ell^2}}{\|w\|_{L^2}}. \tag{3}$$

III. DESIGN OF FRACTIONAL DELAY FILTERS

A. Lifting model of sampled-data error system

Let $\{A, B, C, 0\}$ be a minimal realization of $F(s)$ and $D = mh + d$ where $m \geq 0$ is an integer and $0 \leq d < h$ is a real number. First, we introduce the lifting operator $\mathcal{L}$ [8]:

$$\mathcal{L} : L^2[0,\infty) \ni f \mapsto \{\widetilde{f}[k](\theta)\}_{k=0}^{\infty} \in l^2_{L^2[0,h)}, \ \theta \in [0,h),$$
$$\widetilde{f}[k](\cdot) := f(kh + \cdot) \in L^2[0,h).$$

We apply lifting to the continuous-time signals w and v and put $\widetilde{w} := \mathcal{L}w$, $\widetilde{v} := \mathcal{L}v$. By lifting the continuous-time signals, we obtain a discrete-time system shown in Fig. 3 from the sampled-data system Fig. 2.

Then a state-space realization of the lifted error system shown in Fig. 3 is obtained as follows:

$$\begin{aligned} x[n+1] &= A_d x[n] + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \widetilde{w}[n], \\ e_d[n] &= C_e x[n] - u_d[n], \\ y_d[n] &= C_y x[n], \\ u_d &= K_D y_d, \end{aligned} \tag{4}$$

$$
\begin{aligned}
A_d &:= \begin{bmatrix} e^{Ah} & 0 & 0 \\ Ce^{A(h-d)} & 0 & 0 \\ 0 & B_m & A_m \end{bmatrix}, \\
\mathbf{B}\widetilde{w} &:= \begin{bmatrix} \displaystyle\int_0^h e^{A(h-\tau)} B\widetilde{w}(\tau)d\tau \\ \displaystyle\int_0^{h-d} Ce^{A(h-d-\tau)} B\widetilde{w}(\tau)d\tau \end{bmatrix}, \\
C_e &:= [0, 0, C_m], \quad C_y := [C, 0, 0],
\end{aligned}
$$

where $\{A_m, B_m, C_m, 0\}$ is a realization of discrete-time delay z^{-m}, that is,

$$
\begin{aligned}
A_m &:= \left.\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}\right\} m, \\
B_m &:= [\underbrace{0, \dots, 0}_{m-1}, 1]^T, \\
C_m &:= [1, \underbrace{0, \dots, 0}_{m-1}].
\end{aligned}
$$

B. Norm-equivalent finite dimensional model

The discrete-time state-space realization (4) has an infinite dimensional operator $\mathbf{B} : L^2[0, h) \to \Re^{N+1}$ (N is the dimension of A). By introducing the dual operator $\mathbf{B}^* : \Re^{N+1} \to L^2[0, h)$ of $\mathbf{B}$, we can obtain a finite dimensional operator (i.e., matrix), which leads to a norm-equivalent finite dimensional system of the infinite dimensional system (4). The dual operator $\mathbf{B}^*$ of $\mathbf{B}$ is given as follows [7]:

$$
\begin{aligned}
&\mathbf{B}^* = \begin{bmatrix} \mathbf{B}_1^* & \mathbf{B}_2^* \end{bmatrix}, \\
&\mathbf{B}_1^* := B^T e^{A^T(h-\theta)}, \\
&\mathbf{B}_2^* := \chi_{[0,h-d]}(\theta) B^T e^{A^T(h-d-\theta)} C^T, \\
&\quad \theta \in [0, h], \\
&\chi_{[0,h-d]}(\theta) := \begin{cases} 1, & \theta \in [0, h-d], \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

We have the following lemma:

Lemma 1: The operator $\mathbf{B}\mathbf{B}^*$ is a matrix given by

$$
\begin{aligned}
\mathbf{B}\mathbf{B}^* &= \begin{bmatrix} M(h) & e^{Ad} M(h-d) C^T \\ CM(h-d)e^{A^T d} & CM(h-d)C^T \end{bmatrix}, \\
M(t) &:= \int_0^t e^{A\theta} BB^T e^{A^T\theta} d\theta.
\end{aligned} \tag{5}
$$

Proof: We first prove $\mathbf{B}_1\mathbf{B}_2^* = e^{Ad} M(h-d) C^T$.

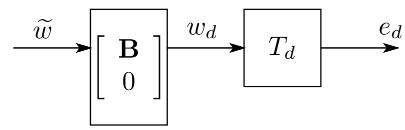

Fig. 4. Sampled-data system $\mathcal{T}_{ew}$

For any $u \in \Re$,

$$
\begin{aligned}
\mathbf{B}_1\mathbf{B}_2^* u &= \int_0^h e^{A(h-\theta)} B \left(\mathbf{B}_2^* u\right) d\theta \\
&= \int_0^h e^{A(h-\theta)} B \left(\chi_{[0,h-d]}(\theta) B^T e^{A^T(h-d-\theta)} C^T u\right) d\theta \\
&= e^{Ad} \int_0^{h-d} e^{A(h-d-\theta)} BB^T e^{A^T(h-d-\theta)} d\theta \, C^T u \\
&= e^{Ad} M(h-d) C^T u.
\end{aligned}
$$

The proof for $\mathbf{B}_1\mathbf{B}_1^*$ and $\mathbf{B}_2\mathbf{B}_2^*$ is similar. ■

Remark 1: It is well known that the matrix $M(t)$ can be computed by using matrix exponential [3]:

$$
\begin{aligned}
&M(t) = F_{22}^T(t) F_{12}(t), \\
&\begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix} := \exp\left\{ \begin{bmatrix} -A & BB^T \\ 0 & A^T \end{bmatrix} t \right\}.
\end{aligned}
$$

By this formula, we can numerically compute the matrix (5) easily.

Using Lemma 1, we obtain a (finite dimensional) discrete-time system T_{ew} which is equivalent to the sampled-data system $\mathcal{T}_{ew}$ with respect to their H^∞-norm.

Theorem 1: For the sampled-data system $\mathcal{T}_{ew}$, there exists a discrete-time system T_{ew} such that

$$
\|\mathcal{T}_{ew}\|_\infty = \|T_{ew}\|_\infty.
$$

Proof: For the state space equations (4), put $w_d := \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \widetilde{w}$ and let T_d denote the system from w_d to e_d (see Fig. 4). Then we have $\mathcal{T}_{ew} = T_d \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$ where T_d is a discrete-time system. From Lemma 1, $\mathbf{B}\mathbf{B}^*$ is finite dimensional. Decomposing $\mathbf{B}\mathbf{B}^*$ as $\mathbf{B}\mathbf{B}^* = B_d B_d^T$ with a

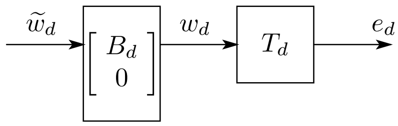

Fig. 5. Norm-equivalent discrete-time system T_{ew}

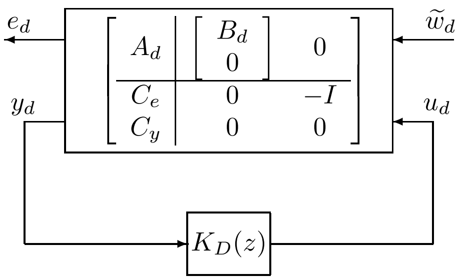

Fig. 6. Norm-equivalent discrete-time system

matrix B_d, we have

$$\begin{aligned}\|\mathcal{T}_{ew}\|_\infty^2 &= \left\| T_d \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \right\|_\infty^2 \\ &= \left\| T_d \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \left(T_d \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \right)^* \right\|_\infty \\ &= \left\| T_d \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}^* T_d^* \right\|_\infty \\ &= \left\| T_d \begin{bmatrix} B_d \\ 0 \end{bmatrix} \begin{bmatrix} B_d \\ 0 \end{bmatrix}^T T_d^* \right\|_\infty \\ &= \left\| T_d \begin{bmatrix} B_d \\ 0 \end{bmatrix} \right\|_\infty^2 .\end{aligned}$$

Putting $T_{ew} := T_d \begin{bmatrix} B_d \\ 0 \end{bmatrix}$ completes the proof. ■

A state space realization of T_{ew} is obtained by replacing $\mathbf{B}$ in (4) with B_d.

Thus the sampled-data H^∞ optimization (Problem 1) is equivalently reduced to a discrete-time H^∞ optimization. The block diagram for the optimization is shown in Fig. 6, and we can easily find the optimal filter $K_D(z)$ by standard softwares such as MATLAB.

IV. DESIGN OF VARIABLE FRACTIONAL DELAY FILTERS

The H^∞ design method in the previous section requires the delay $D = mh + d$ to be fixed. In some applications, a filter with a variable delay (i.e., the delay can be changed without redesigning the filter) is desired in accordance with varying delays. We here give an optimal design for the filter $K_D(z)$ having the delay D (or m and d) as an adaptive parameter.

For this purpose, we assume that the filter $F(s)$ is the first-order low-pass filter:

$$F(s) = \frac{\omega_c}{s + \omega_c}.$$

While this is certainly restrictive, it still has the advantage of having the capability of reconstructing high-frequency components beyond the Nyquist frequency, and the variance from this first-order frequency distribution is estimated by a robustness argument. See Section V below.

We have the following theorem:

Theorem 2: The optimal filter $K_D(z)$ is obtained as follows:

$$\begin{aligned} K_D(z) &= \frac{\omega_c e^{-\omega_c(h-d)} B_0 + (z - e^{-\omega_c h}) B_1(d)}{\omega_c B_0 z^{m+1}}, \\ B_0 &:= \sqrt{\frac{1 - e^{-2\omega_c h}}{2\omega_c}}, \\ B_1(d) &:= \sqrt{\frac{\omega_c}{2}} \cdot \frac{e^{-\omega_c d}(1 - e^{-2(h-d)\omega_c})}{\sqrt{1 - e^{-2\omega_c h}}}. \end{aligned} \tag{6}$$

Moreover, the optimal value of $\|\mathcal{T}_{ew}\|_\infty$ is

$$\|\mathcal{T}_{ew}\|_\infty = B_2(d) := \sqrt{\frac{\omega_c \sinh(\omega_c d) \sinh(\omega_c (h-d))}{\sinh(\omega_c h)}}. \tag{7}$$

Proof: The state space realization of $F(s)$ is $\{-\omega_c, 1, \omega_c, 0\}$. Therefore by using Lemma 1, we have

$$B_d B_d^T = \begin{bmatrix} \dfrac{1 - e^{-2\omega_c h}}{2\omega_c} & \dfrac{e^{-\omega_c d}(1 - e^{-2\omega_c (h-d)})}{2} \\ \dfrac{e^{-\omega_c d}(1 - e^{-2\omega_c (h-d)})}{2} & \dfrac{2(1 - e^{-2\omega_c (h-d)})}{\omega_c} \end{bmatrix}.$$

By the Cholesky factorization of $B_d B_d^T$, we obtain

$$B_d = \begin{bmatrix} B_0 & 0 \\ B_1(d) & B_2(d) \end{bmatrix},$$

where B_0, $B_1(d)$ and $B_2(d)$ are the values in (6) and (7). Then the norm-equivalent discrete-time system (shown in Fig. 6) is given as follows:

$$\begin{aligned} T_{ew}(z) &= (C_e - K_D(z) C_y) \left\{ (zI - A_d)^{-1} \begin{bmatrix} B_d \\ 0 \end{bmatrix} \right\} \\ &= \begin{bmatrix} G_1(z) & G_2(z) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} G_1(z) &:= -K_D(z) C (zI - e^{Ah})^{-1} B_0 \\ &\quad + z^{-m-1} \left\{ C e^{A(h-d)} (zI - e^{Ah})^{-1} B_0 + B_1(d) \right\} \\ &= -K_D(z) \frac{\omega_c B_0}{z - e^{-\omega_c h}} \\ &\quad + z^{-m-1} \left(\frac{\omega_c e^{-\omega_c (h-d)} B_0}{z - e^{-\omega_c h}} + B_1(d) \right), \\ G_2(z) &:= z^{-m-1} B_2(d). \end{aligned}$$

Since $G_2(z)$ is independent of $K_D(z)$, the optimal value is attained by minimizing $G_1(z)$. It is clearly possible to make $G_1(z) = 0$ by taking

$$K_D(z) = \frac{\omega_c e^{-\omega_c(h-d)} B_0 + (z - e^{-\omega_c h}) B_1(d)}{\omega_c B_0 z^{m+1}}.$$

The minimum value of $\|T_{ew}\|_\infty$ is attained as

$$\|T_{ew}\|_\infty = \|G_2(z)\|_\infty = B_2(d).$$

■

Note that the obtained optimal filter consists of a first-order FIR (Finite Impulse Response) filter and discrete-time delay z^{-m}; this is because $F(s)$ is of first order. We conjecture that if $F(s)$ is n-th order, the optimal filter may be the product of an n-th order filter and z^{-m}, since the order of the plant (shown in Fig. 6) is $n + m$.

V. ROBUSTNESS OF DESIGNED FILTERS

Our design requires that the frequency-domain characteristic $F(s)$ be known. In practice, however, $F(s)$ cannot be identified exactly. Therefore, we partially circumvent this defect by discussing the robustness of the filter against the uncertainty of $F(s)$. Let us assume the unstructured uncertainty of the following type:

$$\begin{aligned} F_\Delta(s) &:= F(s)(1 + \Delta(s)), \\ \mathcal{T}_{ew}^\Delta &:= \left(\mathcal{S}_h e^{-Ds} - K_D \mathcal{S}_h\right) F_\Delta, \\ \Delta \in \boldsymbol{\Delta} &:= \left\{\Delta : \|1 + \Delta\|_\infty \le \gamma\right\}. \end{aligned}$$

We have the following proposition:

Proposition 1: For any $\Delta \in \boldsymbol{\Delta}$,

$$\|\mathcal{T}_{ew}^\Delta\|_\infty \le \gamma \|\mathcal{T}_{ew}\|_\infty \tag{8}$$

Proof: By the submultiplicativity of H^∞-norm, we readily have

$$\begin{aligned} \|\mathcal{T}_{ew}^\Delta\|_\infty = \|\mathcal{T}_{ew}(1 + \Delta)\|_\infty &\le \|\mathcal{T}_{ew}\|_\infty \|1 + \Delta\|_\infty \\ &\le \gamma \|\mathcal{T}_{ew}\|_\infty. \end{aligned}$$

■

By this proposition, the nominal performance $\|\mathcal{T}_{ew}\|_\infty$ is guaranteed against the perturbation $\Delta \in \boldsymbol{\Delta}$ if $\gamma \le 1$. In some cases, it is possible that $\gamma = 1$, in which case the performance is bounded as illustrated in Fig. 7. This means that if we take $F(s)$ that covers all possible gain characteristics of the input analog signals, it gives a bound for the error norm. This at least partially justifies the choice of the first-order weighting $F(s)$ in the previous section.

VI. DESIGN EXAMPLE

We present a design example of fractional delay filters. The design parameters are as follows: the sampling period $h = 1$, the delay is $D = 10.8$, that is, $m = 10$ and $d = 0.8$, and

$$F(s) = \frac{0.5}{s + 0.5}.$$

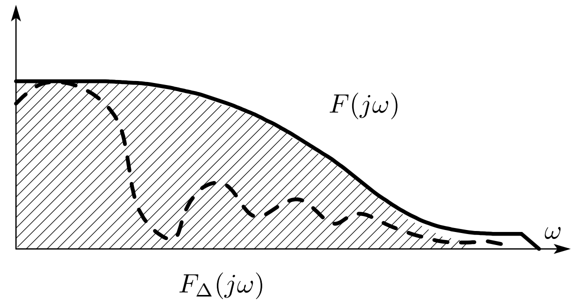

Fig. 7. Nominal filter $F(s)$ (solid) and perturbed $F_\Delta(s)$ (dash)

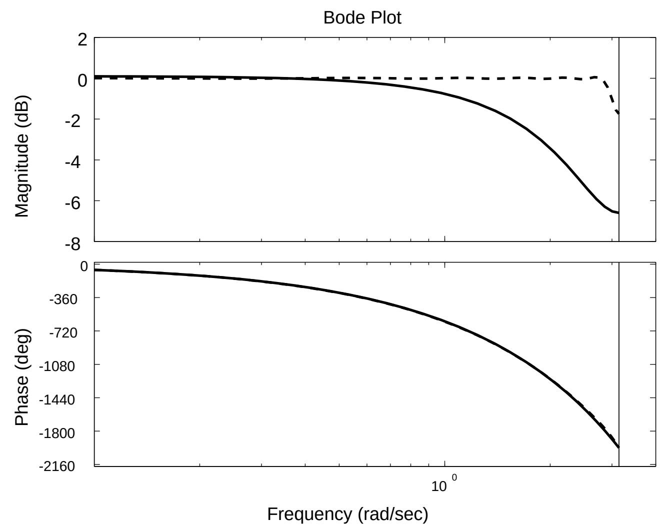

Fig. 8. Bode plot of filters: sampled-data design (solid) and conventional design (dash)

Note that $F(s)$ has the cutoff frequency $\omega_c = 0.5$.

We compare the optimal filter obtained by Theorem 2 with a conventional filter of 31-st order FIR filter with the Kaiser window [2].

Fig. 8 shows the Bode plots of the filters obtained by the sampled-data design and the conventional one. As illustrated in Fig. 8, the conventional filter is closer to the ideal filter (2) as expected, so that it appears better in the context of the conventional design methodology.

However, the present filter exhibits much smaller errors in the high-frequency domain as shown in Fig. 9. This is because the conventional design does not take into account the frequency response of the source analog signals while the present method does.

To see the difference between the present filter and the conventional one, we show the time response against rectangular waves in Fig. 10 (sampled-data design) and in Fig. 11 (conventional design). The present method is clearly superior to the conventional one which shows much ringing at the edges of the waves.

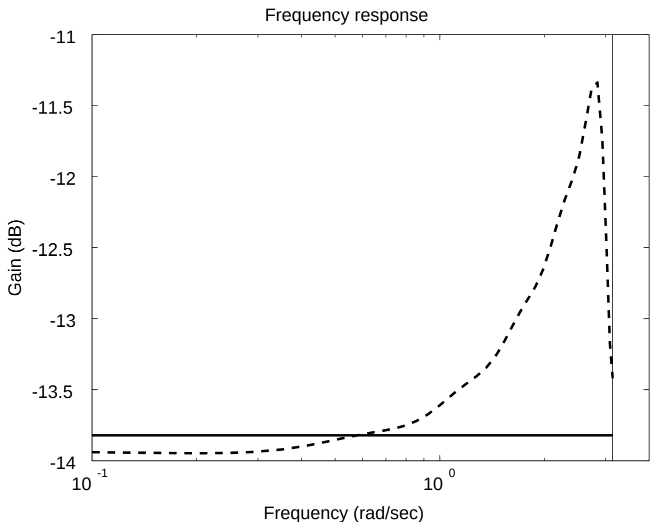

Fig. 9. Frequency response of $\mathcal{T}_{ew}$: sampled-data design (solid) and conventional design (dash)

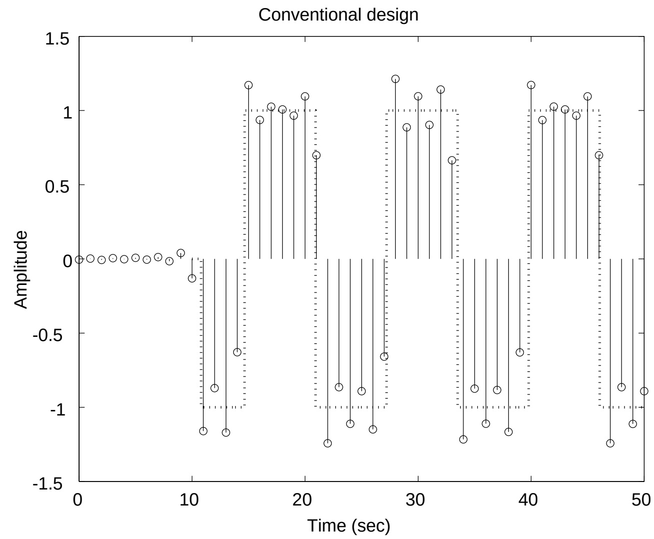

Fig. 11. Time response (conventional design)

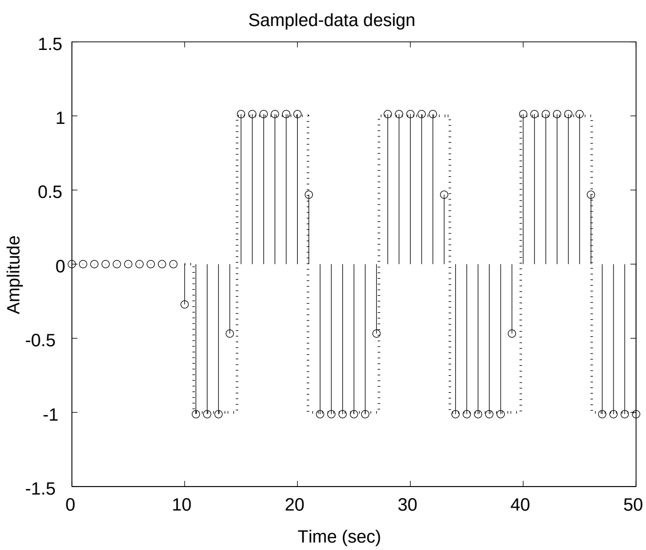

Fig. 10. Time response (sampled-data design)

VII. CONCLUSION

We have presented a new method of designing fractional delay filters via sampled-data H^∞ optimization. An advantage here is that an analog optimal performance can be obtained. We have also given the H^∞ optimal filter having delay time variable D as an adaptive variable, when the frequency distribution of the input analog signal is modeled as a first-order low-pass filter. While the hypothesis here is somewhat restrictive, we have also shown that a certain error bound can be obtained for a more general class of weights. The designed filter exhibits a much more satisfactory performance than conventional ones.

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