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Approximating Sampled-Data Systems with Applications to Digital Redesign

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Abstract

Despite the existence of methods for a direct optimal design of sampled-data control systems, it is often desired to approximate sampled-data systems with discrete-time ones. This occurs frequently in the context of digital redesign, in which one retains intuition in the continuous-time context. This paper investigates and gives conditions under which such a method works. Under some conditions, we guarantee that fast-sampling approximation works well for H^∞ sampled-data design, and give an estimate for such an upsampling factor. As an application, we propose a new method for obtaining an FIR controller (possibly with first-order approximation). A comparison is made with an existing method.

1 Introduction

Modern sampled-data control theory enables us to design directly a digital controller that makes an analog performance optimal. It is also known that for finite-dimensional plants this design problem, for example, H^∞ control problem having the mixed nature of both discrete-time and continuous-time, can be reduced to an equivalent discrete-time, finite-dimensional problem [3, 6, 7].

In spite of these results, however, there are still several reasons that lead us to approximate the design problem in a variety of ways by a more conventional approximate design problem, and then obtain a digital controller. One reason is that we may be able to use a design software of our choice, and base our design intuition on such softwares we are familiar with. Once we have obtained an approximant, it is not difficult to invoke such a more conventional design package. Yet another advantage is that we may wish to rely on some design intuition we have developed, and this is in many cases easier to maintain by resorting to approximations, rather than using the direct optimization methods mentioned above. So-called *digital redesign* is regarded as such a case: we first obtain a continuous-time controller, and then attempt to discretize it, while attempting to maintain the desirable continuous-time performance.

In such attempts, we of course wish to capture the continuous-time performance, rather than the mere sample-point behavior. This requires approximation of the generalized plant in the sampled-data setting, and desirably a convergence analysis. Keller and Anderson [1, 8] have proposed to use a fast-sampling/fast-hold approximation with its approximating performance taken to be the closed-loop behavior. Related design problems have been studied by Madievski and Anderson [10] for H^∞ control and by Bamieh et al. [4] for L^1 control problems. Later it was proven by Yamamoto et al. [14] that such fast-sampling approximations converge uniformly in the frequency domain to the limiting sampled-data system. The following important issues still remain open, however:

1. How do we guarantee that a designed controller K (based on plant approximation/discretization) converges to the "right" controller in the limit? Since we do *not* know in advance which controller will be part of the closed-loop before the design, a convergence theorem with a *fixed* K would not be enough. We need some kind of uniformity in the controllers K in the convergence process to guarantee this. (While Bamieh et al. [4] successfully gave an explicit convergence bound for the L^1 control design problem, there is an important difference here. Although sampling is continuous with respect to the L^∞ norm where it is well defined, it is not continuous with respect to the L^2 norm. This makes the estimate of the convergence rate here in the H^∞ context more delicate, and quite different from that studied in [4].)

2. How fast should fast-sampling periods be? While there exist some empirical studies on estimates and indications in some special cases, there does not exist a precise estimate that guarantees quality of approximation for a particular rate, to the best of the authors' knowledge (again, except [4] for the L^1 control problem, but not for the frequency response computation/H^∞ design).

This paper studies these problems. We first invoke the so-called FR-operator $\mathcal{T}$ by Araki and others [2]. Its principal submatrix $\mathcal{T}(M, M)$ of size M tells us how many aliased components we need to obtain a desired degree of precision. We then obtain an estimate on the fast-sampling rate using $\mathcal{T}(M, M)$, based on the approximation estimate of sinusoids via piecewise step functions arising from sample and hold actions.

We also apply the results to the digital redesign problem, and show that the FIR (finite-impulse response, i.e., those

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having finitely many Markov parameters) approximation problem of a stable controller for sampled-data systems can be reduced (with first-order approximation when the original problem is not one-block) to an LMI. In the field of digital filters, there is always a strong need for replacing an IIR (infinite-impulse response) filter by an FIR filter. The latter has the advantages that it is intrinsically stable and free from various problems such as limit cycles arising from finite-wordlength precision. We show that the obtained LMI can be solved via a convex optimization method such as the cutting plane method. We illustrate the result by comparing it with the example studied in [10].

2 Uniform Approximation via Fast-Sampling

Consider the sampled-data control system in Fig. 1, consisting of a continuous-time generalized plant $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and a discrete-time controller K with sampler $\mathcal{S}_h$ and zero-order hold $\mathcal{H}_h$ with sampling period h. The closed-loop transfer operator from w to z will be denoted by $\mathcal{T}_{zw}(K)(z)$.

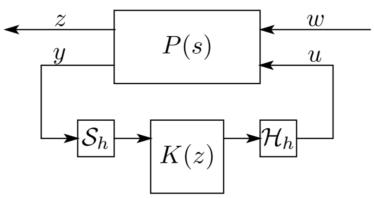

Figure 1: Sampled-data control system

We assume that P_{21} is strictly proper, which is necessary to assure that sampling is well-defined. Likewise, we assume that P_{11} and P_{12} are also strictly proper.

It is well known that for a finite-dimensional plant P, the sampled-data H^∞ controller design problem of Fig. 1 can be reduced to a norm-equivalent discrete-time H^∞ design problem [3, 6, 7]. Nonetheless, discretization via fast-sampling and fast-hold as studied in [10, 14] is often effective as explained in the introduction.

Let us thus consider fast-sampling approximants for the design problem (The N-step closed-loop transfer operator from w to z will be denoted by $\mathcal{T}^N_{zw}(K)(z)$.) Fig. 2; for detailed formulae, see, e.g., [14]. The idea is to approximate the inputs by step functions of step size h/N for sufficiently large N and also approximate outputs by taking their samples at every h/N seconds rather than the original sampling period h. It may appear trivial that such an approximation gives the right answer, for example, in computing the frequency response. The situation is however much more delicate because we do not know in advance what kind of input gives rise to the gain of the frequency response which is actually the operator norm at each frequency. It is proven in [14] that the frequency response gain of such approximants converges to that of the limit uniformly for $0 \le \omega \le 2\pi/h$, *for a fixed* controller K.

While this is sufficient for analysis purposes, we still need to go one step further. In synthesis, we do not know in advance which controller will appear in the closed-loop. The con-

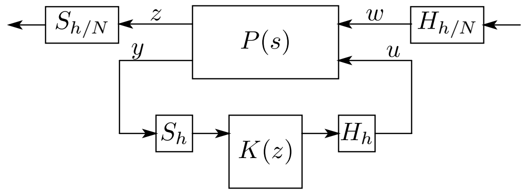

Figure 2: Fast sampling approximation

vergence result of [14] requires that the controller be fixed, and this assumption is (in effect) not satisfied for synthesis problems. To this end, we must guarantee the *uniform* convergence, with respect to controller K, of the norms of the fast-sampling approximants if we are going to use such approximants in designing the controller. This motivates the following theorem.

Theorem 2.1 *Let S be a set of stable controllers K such that i) every $K \in S$ is stabilizing, and ii) S is compact with respect to the H^∞-norm. Then the frequency response gain $\|\mathcal{T}^n_{zw}(K)(e^{j\omega h})\|$ of the n-step fast-sampling approximant $\mathcal{T}^n_{zw}(K)$ converges to $\|\mathcal{T}_{zw}(K)(e^{j\omega h})\|$ uniformly in $K \in S$. This convergence is also uniform in $\omega \in [0, 2\pi/h)$.*

Proof Fix $\epsilon > 0$, and take any $K \in S$. By the convergence result of [14], there exists N such that

$$\left| \|\mathcal{T}^n_{zw}(K)(e^{j\omega h})\| - \|\mathcal{T}_{zw}(K)(e^{j\omega h})\| \right| < \epsilon \text{ for all } n \ge N,$$

and this is uniform in ω. We will thus omit the dependence on ω below. Take the least such N and name it $N(K, \epsilon)$. Since $\|\mathcal{T}^n_{zw}(K)\| - \|\mathcal{T}_{zw}(K)\|$ is continuous with respect to K as Lemma 2.3 below shows, there exists a neighborhood $B(K, \delta) := \{K' : \|K' - K\| < \delta\}$ of K such that

$$\left| \|\mathcal{T}^n_{zw}(K')\| - \|\mathcal{T}_{zw}(K')\| \right| < \epsilon$$

for all $n \ge N(K, \epsilon)$ and $K' \in B(K, \delta)$. This yields a covering of the controller set:

$$S = \cup_{K \in S} B(K, \delta).$$

By the compactness of S, there exists a subcovering

$$S = B(K_1, \delta_1) \cup \cdots \cup B(K_m, \delta_m).$$

Taking $N_{\max} := \{N(K_1, \epsilon), \ldots, N(K_m, \epsilon)\}$, we readily have that $n \ge N_{\max}$ implies

$$\left| \|\mathcal{T}^n_{zw}(K)\| - \|\mathcal{T}_{zw}(K)\| \right| < \epsilon \qquad (1)$$

for all $K \in S$. $\square$

Remark 2.2 A typical example of S may be given by the set of FIR controllers of order less than or equal to a prefixed integer q and with a prespecified norm bound as

$$S := \{K : \operatorname{ord} K \le q, \|K\| \le M, K \text{ is stabilizing}\}.$$

Since the number of coefficients of K is at most $q + 1$, this can be regarded as a subset of $\mathbb{R}^{q+1}$. The norm $\|K\|$ may be interpreted in the sense of any norm in $\mathbb{R}^{q+1}$. More generally, one can also consider IIR controllers, having both numerator and denominator, with fixed order. To avoid unstable pole-zero cancellations, we may have to exclude a small open neighborhood around unstable poles in considering S.

Theorem 2.1 states that if we agree to fix a compact set S and search our controller K in this set, there exists N for a given ϵ such that if $\|\mathcal{T}_{zw}^N(K)\| < \gamma$ then $\|\mathcal{T}_{zw}(K)\| < \gamma+\epsilon$. That is, a controller designed by the N-step approximation indeed approximates the performance of the solution to the original problem without approximation. Furthermore, if $\|\mathcal{T}_{zw}(K)\| < \gamma$ and if K is known to be in S, then it can be found with error tolerance ϵ by taking N large enough.

It remains to show the following lemma on the continuity of the closed-loop operator with respect to K.

Lemma 2.3 *Consider the closed-loop operator $\mathcal{T}_{zw}(K)$ where K is assumed to be stable and stabilizing. Then $\mathcal{T}_{zw}(K)$ is continuous in K with respect to the H^∞ norm.*

Sketch of Proof Let $\|K' - K\|_\infty < \delta$ with K, K' stabilizing. Put $\Delta := K' - K$, and we can easily rewrite $\mathcal{T}_{zw}(K')$ involving Δ and the nominal closed-loop operator $\mathcal{T}_{zw}(K)$. Then for sufficiently small δ, $\|\Delta\|_\infty < \epsilon$ enables us to invoke the small-gain theorem, and the result follows. □

3 Estimating the Upsampling Factor N

Theorem 2.1 above guarantees that we can rely on fast-sampling approximations for designing a sampled-data H^∞ stable controller, by taking the upsampling factor N sufficiently large. Our next question is naturally *how large such an N should be*.

To the best of our knowledge this question has not been investigated in depth in the literature even for the analysis problems (except [4] for L^1 control problem). We give an answer to this by invoking certain a priori bounds and estimates for plants and controllers. Consider Fig. 1 again.

Let $\mathcal{T}_{zw}(e^{j\omega h})$ denote the frequency response operator associated with the closed-loop transfer operator $\mathcal{T}_{zw}$. We will omit the dependence on K hereafter. This operator is also equivalent [13] to the notion of the so-called FR operator [2]. We denote this operator by $\mathcal{T}(\omega)$. In essence, $\mathcal{T}(\omega)$ is nothing but the (infinite) matrix representation of $\mathcal{T}_{zw}(e^{j\omega h})$ with respect to an orthonormal basis $\{\exp(j(\omega + 2n\pi/h)t\}_{n=-\infty}^{\infty}$ in $L^2[0, h]$ [13].

Let us first note the following formula for $\mathcal{T}$.

Proposition 3.1 *Let P_{ij}, $\mathcal{T}_{zw}$ and $\mathcal{T}(\omega)$ be as above, and suppose that the closed-loop system is stable. Let P_{22}^* denote the pulse-transfer function associated with P_{22}, and let $V := (I - P_{22}^* K)^{-1}$. Then the (i, k) entry $\mathcal{T}_{ik}(\omega)$ of $\mathcal{T}$ is given by*

$$\mathcal{T}_{ik}(\omega) = P_{11}(j\omega_i) + \frac{1}{h}P_{12}(j\omega_i)\hat{H}(j\omega_i)K(e^{j\omega h})V(e^{j\omega h})P_{21}(j\omega_k) \tag{2}$$

Outline of Proof Follow the proof in Yamamoto and Araki [13] given for the unity feedback case. □

3.1 A Special Case

In what follows we first make the following assumptions:

- **(A)** $P_{11} = 0$.
- **(B)** Each P_{ij} is an SISO system.
- **(C)** $|P_{21}(j\omega)|$ is rational and monotone decreasing for $\omega > 0$.

The first assumption is somewhat specialized, but it greatly simplifies the subsequent arguments. We will discuss a generalization in the next subsection. The second assumption is just for convenience of notation, and it does not lead to any real loss of generality. The third assumption is also placed to simplify the subsequent arguments. If this is not satisfied by a possible peak in the high frequency region, then we should simply replace it by a suitable SISO anti-aliasing filter to estimate the norm of $\mathcal{T}$.

Denote by $\mathcal{T}(L, M)$ the principal submatrix $\{\mathcal{T}_{ij}\}_{0\le i\le L, 0\le j\le M}$ of $\mathcal{T}$. We allow L, M to be ∞ as a special case, in which case the respective $\le$ in the indices should be replaced by $<$.

Let us first note the following:

Lemma 3.2 *Assume (A)–(C) above, and let $C_0 := \sup_{0\le\omega\le 2\pi/h} \sqrt{\sum_{n=1}^{\infty} |P_{21}(j\omega_k)/P_{21}(j\omega)|^2}$, where $\omega_k := \omega + 2k\pi/h$. By assumption (C), this quantity is finite. Then*

$$\|\mathcal{T}\| \le C_0 \|\mathcal{T}(\infty, 0)\|. \tag{3}$$

In other words, only the base-band inputs are enough to estimate the norm of $\mathcal{T}$.

Proof Let $e^{j\omega_k t}$ be the k-th aliased input. When it is applied to the system, it first goes through $P_{21}(s)$ with steady-state output $P_{21}(j\omega_k)e^{j\omega_k t}$. After sampling, this becomes a discrete-time signal $\{P_{21}(j\omega_k)e^{j\omega\ell h}\}_{\ell=0}^{\infty}$. Hence the steady-state output corresponding to the input $\sum \alpha_k e^{j\omega_k t}$ is also realized by an input $(\sum_k \alpha_k P(j\omega_k)/P(j\omega))e^{j\omega t}$. Since $|\sum_k \alpha_k P(j\omega_k)/P(j\omega)| \le C_0\sqrt{\sum_k |\alpha_k|^2}$ by Schwarz's inequality, the norm of $\mathcal{T}$ is easily seen to be bounded by the right-hand side of (3). □

Remark 3.3 Since $|P_{21}(j\omega_k)/P_{21}(j\omega)|$ decays by arithmetic progression, C_0 is roughly estimated by $\pi/\sqrt{6}$.

Let us next investigate how large M for $\mathcal{T}(M, 0)$ should be taken to estimate $\mathcal{T}(\infty, 0)$.

Proposition 3.4 *Given $\epsilon > 0$, there exists M such that*

$$|\mathcal{T}_{i0}| \le \frac{\epsilon}{C_0\pi/\sqrt{6}}, \quad |i| > M. \tag{4}$$

This in turn yields the estimate

$$\|\mathcal{T} - \mathcal{T}(M, 0)\| < \epsilon. \tag{5}$$

Proof Note first that $\hat{H}(j\omega) = (1 - e^{-j\omega h})/j\omega$ is strictly proper and monotone decreasing. Hence (2) and assumption (A) imply the existence of M.

To compute the induced norm $\mathcal{T} - \mathcal{T}(M, 0)$, note first that as in Lemma 3.2

$$\|\mathcal{T} - \mathcal{T}(M, 0)\| \le C_0 \|\mathcal{T}(\infty, 0) - \mathcal{T}(M, 0)\| \quad (6)$$

holds. Since $\mathcal{T}(\infty, 0) - \mathcal{T}(M, 0)$ is a single column vector, we have, by the Schwarz inequality,

$$\begin{aligned} \|\mathcal{T}(\infty, 0) - \mathcal{T}(M, 0)\|^2 &\le \sum_{i>M} |\mathcal{T}_{i0}|^2 \\ &\le \sum_{i>M} \frac{\epsilon^2}{|i - M|^2} = \pi^2 \epsilon^2 / 6. \end{aligned}$$

This, along with (6), implies (5) □

Remark 3.5 If P_{12} is strictly proper with relative degree r, then the last estimate can be improved to $\sum_{i>M} \epsilon^2 / |i - M|^{2(r+1)}$.

Remark 3.6 Suppose $P_{12} = 1/(1 + Ts)$ with $1/T \sim \pi/2h$, i.e., the bandwidth is about half the Nyuqist rate. This implies that $M = 3$ means the bandwidth $6\pi/h > 5\pi/h = 10 \times \pi/2h = 10/T$, i.e., 1 decade. Hence it rougly decays at least by -20dB. More generally, if P_{12} is strictly proper, then it decays by order 2, and hence $M = 3$ guarantees ϵ to be of order 10^{-2}.

Proposition 3.4 tells us how many blocks we should take to compute $\mathcal{T}$. This in turn means that we need sinusoids $e^{j\omega t}$ up to the frequency $\omega = 2M\pi/h$ to attain the desired accuracy as specified in this proposition. The rest of the work is then to estimate the upsampling factor N to obtain the accuracy of (4). We have the following estimate for a typical case.

Proposition 3.7 *Assume* $\sup_{0 \le \omega \le 2\pi/h} |P_{21}(j\omega)| = 1$ *for simplicity, and put* $C(z) := K(z)V(z)$, $\psi(\omega) := |1 - e^{j\omega h/N}|$. *Suppose also that* $|P_{12}(j\omega)|$ *is bounded by* $1/\sqrt{1 + \omega^2 T^2}$ *for some first order plant* $1/(1 + Ts)$ *with* $1/T \approx \pi/h$. *Likewise, we also assume* $|C(e^{j\omega h})|$ *and* $|P_{21}(j\omega)|$ *are also bounded by* $1/\sqrt{1 + \omega^2 T^2}$ *at least up to the Nyquist frequency* π/h. *Then* $|\mathcal{T}_{i0}(\omega)|$ *decays by* -40*dB/dec, and attains the maximum error level of* 0.022 *when* $N = 5$.

Outline of Proof By the assumptions,

$$|\mathcal{T}_{i0}(\omega)| \le \left| P_{12}(j\omega_i)(\hat{H}(j\omega)/h) C(e^{j\omega h}) P_{21}(j\omega) \right|.$$

at each frequency. Sampling $e^{j\omega t}$ at $t = kh/N$, $k = 0, 1, 2, \ldots$ induces the error $\psi(\omega) e^{j\omega t}$ which is bounded by $\psi(\omega)$. Then the total error in $|\mathcal{T}_{i0}(\omega)|$ induced in fast sampling is bounded by

$$\psi(\omega) |P_{12}(j\omega)(\hat{H}(j\omega)/h) C(e^{j\omega h}) P_{21}(j\omega)|. \quad (7)$$

Here $|\hat{H}(j\omega)/h| = |(1 - e^{j\omega h})/j\omega h|$ is bounded by $\min\{1, 2/\omega\}$, which decays at -20dB/dec in the high frequency range. The same can be said of $|P_{21}(j\omega)|$ in the high frequency range. Furthermore, when close to the Nyquist frequency, both $|C(e^{j\omega h})|$ and $|P_{21}(j\omega)|$ are bounded by $1/\sqrt{1 + \omega^2 T^2}$, so that they contribute to the decay of $\mathcal{T}_{i0}(\omega)$ at least up to the Nyquist frequency. On the other hand, $\psi(\omega)$ governs the error in the low frequency range, and this is determined by the upsampling factor N. It is easy to compute the order of $\psi(\omega)$ and see that it is of order $\omega h/N$ for small ω (by Taylor expansion), and bounded by 2 for large ω (note $|e^{j\omega h/N}| = 1$). For $N = 5$, this is about 0.62 at the Nyquist frequency, and increases monotonically up to $N\pi/h$rad/sec. But then the high-frequency roll-off of $\hat{H}(j\omega)/h$ becomes prevalent, and the overall product decays approximately by -40dB/dec. A numerical computation exhibits that the overall maximum is attained at around 1rad/sec with value of 0.022 in gain (i.e., about -33dB). □

Remark 3.8 The above estimate is overly conservative. This conservativeness arises from using $\psi(\omega)$ for the sampling error. While it is certainly correct for a *single* sinusoid, we should note that the gain of $\mathcal{T}(\omega)$ is obtained as the response against the *worst-case* input. Thus, even for each frequency, it is possible to consider a piecewise constant input, different from the one obtained by sampling the sinusoid, but giving rise to less error. In fact, just by shifting the sampled input by half the upsampling step size, one achieves a substantially small error in L^2-norm. For an input $e^{j\omega t}$, this is just a phase shift (advance) by $h/2N$, and is achieved by multiplying $e^{jh/2N}$. The resulting sampled output has phase advance by $h/2N$ against the output resulting from $e^{j\omega t}$. In the case of the proposition above with $N = 5$, the overall L^2 error in one upsampled step h/N is at the level of 0.007—approximately $1/10$ of the error estimated by $\psi(\omega)$. This explains the true reason why the fast-sampling approximation works so well in computation of the frequency response and H^∞ designs for sampled-data systems. (In fact, even $N = 3$ works in the case above.)

We now turn our attention to estimating the approximation level in the fast-hold in the input term. In view of Lemma 3.2, it is enough to estimate $\mathcal{T}(\infty, 0)$. This means we need only consider the single sinusoid $e^{j\omega t}$ for each ω. If we approximate this by the fast-hold of step N, there will again be an error proportional to $\psi(\omega)$ above. However, what is different here is that we can arbitrarily change the fast-hold input function $u_N(t)$ to attain the norm of the operator. In particular, we can change it so that the sampled values after going through $P_{21}(s)$ at the original sampling period match those derived from input $e^{j\omega t}$.

We start with the following:

Lemma 3.9 *Suppose there exists* $T > 0$ *such that*

$$|P_{21}(j\omega)| \le \left| \frac{1}{1 + jT\omega} \right| \quad (8)$$

Let $\mathcal{V}$ *denote the operator obtained by replacing* $P_{21}(s)$ *by* $1/(1 + Ts)$ *in* $\mathcal{T}(\infty, 0)$. *Then the error in the norm of* $\mathcal{T}(\infty, 0)$ *induced by the fast-hold* $H_{h/N}$ *is less than that of* V *induced by* $H_{h/N}$.

Proof Since $|P_{21}|$ is assumed to be monotone decreasing, (8) is not restrictive.

Note that in view of (2) $\mathcal{T}(\infty, 0)$ can be decomposed as $\mathcal{T}(\infty, 0) = \Gamma P_{21}$ for some $Gamma$. Then we have

$$\begin{aligned} \|\mathcal{T}(\infty,0)(\omega)\| &= |P_{21}(j\omega)|\|\Gamma(j\omega)\| \\ &\leq |1/(1+jT\omega)|\|\Gamma(j\omega)\| \\ &= \|1/(1+jT\omega)\Gamma(j\omega)\| = \|V(\omega)\|. \end{aligned}$$

This means that the error arising from fast-hold $H_{h/N}$ can be bounded by that of V. □

This lemma reduces the problem to the simple case $P_{21}(s) = 1/(1+Ts)$.

Lemma 3.10 *Let $P_{21}(s) = 1/(1+Ts)$, and the sampling period h not pathological, i.e., the zero-order hold equivalent of P_{21} be minimal. Then for any $N \geq 1$, and for any input $u = e^{j\omega t}$, there exists a piecewise constant input v (with step length h/N) with the same norm as u such that the sampled output value of P_{21} at $kh, k = 0, 1, 2, \ldots$ are identical to those produced by u.*

Proof We prove that starting from the zero initial state, it is possible to find v with the same norm as u such that at $t = h$ the attained state is equal to that produced by input u. Observe first that without the norm constraint there is always such a v because the zero-order hold equivalent system remains reachable by hypothesis, and $N \geq 1$. Thus we need only to guarantee that the minimum norm among such v does not exceed that of u. This is governed by the norm of the reachability (controllability) Grammian

$$\int_0^h e^{At}BB^Te^{A^Tt}dt \tag{9}$$

or by

$$\sum_{k=0}^{N}(e^{Ah/N})^k\tilde{B}\tilde{B}^T(e^{A^Th/N})^k \tag{10}$$

for the hold-equivalent system, where (A, B, C) is a minimal realization of P_{21} and $\tilde{B} = \int_0^{h/N} e^{At}Bdt$. It is routine to check that (10) coincides with (9) for a first-order system.

Then the least norm of such v does not exceed that of the least norm of a continuous-time input attaining the same final state at $t = h$. They by lifting this piecewise constant input in the period h, we obtain an input which gives rise to the same sampled values at $t = kh, k = 0, 1, 2, \ldots$. □

This lemma guarantees that the fast-hold device does not deteriorate the norm estimate of $\mathcal{T}$. This is a consequence of the fact that we look for the worst-case inputs (hence the intermediate behavior of the input in fast-hold need not be the same as the original sinusoid $e^{j\omega t}$).

3.2 Outline of the general case $P_{11} \neq 0$

When $P_{11} \neq 0$, the argument above applies to the estimate on the effect of sampling of outputs. The difference is that we cannot count exclusively on the effect of the hold $\hat{H}(s)$ and the discrete-time controller for the high-frequency roll-off, so that it is reasonable to assume -20 or -40dB/dec decay for P_{11}. But aside from this, the argument in Remark 3.8 applies, so the error in the low frequency range can be expected to be very low, even for relatively small N.

On the other hand, since there is no sampling in the input term, we cannot invoke Lemma 3.10 for estimating the error arising from the fast-hold inputs. This requires further study, but at least, the worst-input analysis as given in Remark 3.8 will help ensure that the convergence is much faster than might be normally expected.

4 Digital Redesign via Fast-Sampling

In this section, we treat a digital redesign problem by using the fast-sampling method. We will show that the problem reduces to a convex optimization problem.

Consider the continuous-time feedback system shown in Fig. 3. In the figure, P is a continuous-time plant and K is a continuous-time controller.

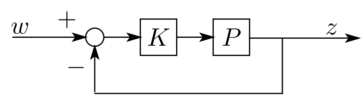

Figure 3: Continuous-time feedback system

Many methods are known for designing such a controller, while the implementation often needs a discrete-time controller. Therefore the feedback system may be realized as a sampled-data system shown in Figure 4. In the figure,

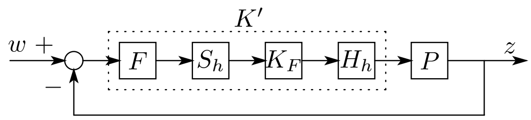

Figure 4: Sampled-data feedback system

F is an anti-aliasing filter for sampling which is stable and strictly proper.

Then our problem is stated as follows:

Problem 1 *Given a stable continuous-time controller $K(s)$ which stabilizes the feedback system in Fig. 3, find an FIR controller K_F which approximates the performance of the continuous-time system.*

To achieve a good approximation, we optimize the H^∞ error between the systems in Fig. 3 and Fig. 4. Let $K' := H_hK_FS_hF$ and $\mathcal{T}_{zw}(\cdot)$ denote the transfer operator from w to z depending on the situation, e.g. $\mathcal{T}_{zw}(K) := PK(I+PK)^{-1}$. Then we express the problem above as an H^∞ optimization problem:

Problem 2 *Given $\gamma > 0$, find an FIR controller K_F which satisfies*

$$\|\mathcal{T}_{zw}(K) - \mathcal{T}_{zw}(K')\|_\infty < \gamma \tag{11}$$

In order to make the problem a convex optimization, we first apply a first-order approximation in $K - K'$ to the error system

$$\mathcal{T}_{zw}(K) - \mathcal{T}_{zw}(K') \approx W(K - K')V$$

where $W := (I+PK)^{-1}P$ and $V := (I+PK)^{-1}$. Therefore the sampled-data error constraint (11) can be converted to the error constraint

$$\|W(K-K')V\|_\infty < \gamma. \tag{12}$$

Then we apply the fast-sampling approximation to the sampled-data H^∞ optimization problem (12), which can be a discrete-time H^∞ optimization:

$$\|G_1(z) - G_2(z)K_F(z)G_3(z)\|_\infty < \gamma \tag{13}$$

where G_1, G_2 and G_3 are fast-sampling approximations of $W(s)K(s)V(s)$, $W(s)H_h$ and $S_hF(s)V(s)$ respectively. Let the FIR controller $K_F := \sum_{k=0}^{N-1} \alpha_k z^{-k}$. Since $G_1 + G_2K_FG_3$ is affine in K_F (or the parameters α_k), the problem of minimizing the left side of (13) can be expressed as a convex optimization, which can be easily solved by some methods, e.g. the LMI or the cutting-plane method [5].

5 Design Example

In this section, we study the example given in [10]. The upsampling factor N for fast-sampling is 5, and the order of the FIR controller is 63. For comparison, the following controllers are also given:

- H^∞-optimal sampled-data IIR controller (designed by the fast-sampling approximation).
- reduced order IIR controller of the continuous-time H^∞-optimal controller obtained by frequency-weighted balanced truncation [10].

Fig. 5 shows the gain comparison of the error $\mathcal{T}_{zw}(K) - \mathcal{T}_{zw}(K')$ (see (11)) for the optimal FIR, the optimal IIR and the balanced truncation. It can be seen that the maximum gain of the error (i.e. H^∞ norm) with FIR is better than that with the balanced controller.

To illustrate the difference in the performance with these controllers, we show the gain of the sampled-data feedback system $\mathcal{T}_{zw}(K')$ and the original continuous-time $\mathcal{T}_{zw}(K)$ in Fig. 6. In the figure, we can see that the system with the balanced controller has a peak around $\omega = 0.8$ [rad/sec]. On the other hand, the FIR controller does not show such a peak although it shows larger errors in the high frequency range.

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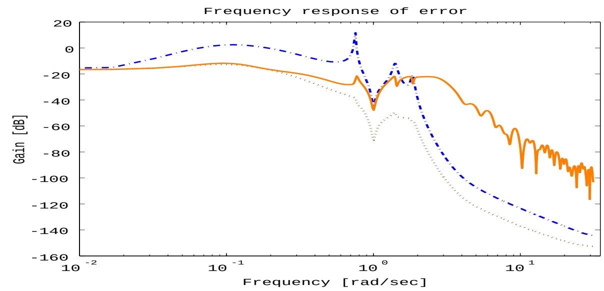

Figure 5: Gain of Error $\mathcal{T}_{zw}(K) - \mathcal{T}_{zw}(K')$: optimal FIR (solid), optimal IIR (dot) and balanced truncation (dash)

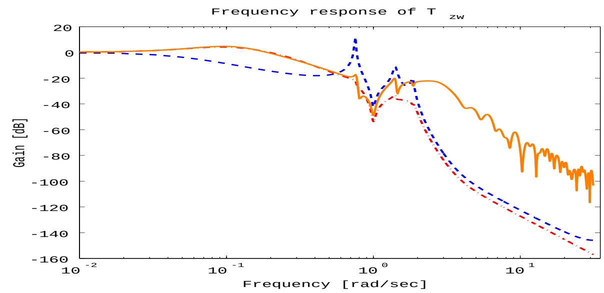

Figure 6: Gain of $\mathcal{T}_{zw}(K')$: optimal FIR (solid) and balanced truncation (dash), and original $\mathcal{T}_{zw}(K)$(dash-dot)

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