A New Design for Sample-Rate Converters

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Abstract

This paper proposes a new design methodology for digital filters, interpolators/decimators and sample-rate converters, based on the modern sampled-data control theory. In contrast to the conventional filter designs where the methods are mostly based on frequency domain approximation techniques, the proposed method makes use of the sampled-data H^∞ control theory, thereby allowing for optimizing the intersample behavior and aliasing effects. The novel feature here is that the proposed method can optimize the analog-domain performance over all frequency ranges, thereby guaranteeing a desirable performance without breaking the design problem into several different steps, such as linear phase characteristic, optimal attenuation level design, etc. A design example is presented to show the advantages of the present method.

1 Introduction

Digital filter design is, in a sense, an art of approximation which takes many different specifications into account in several different steps: linear phase shift property, smooth pass-band transmission, high attenuation level in the stop band, desirable transition band characteristic, etc. Many guiding quantities have been introduced to help the designer [5, 9, 10].

However, such a design can be quite complicated, and require trained skills. It is furthermore executed mostly in the discrete-time domain. The continuous-time performance is indirectly discussed via the notion of aliasing.

One may however note that, in many applications, the performance we wish to optimize is still in the analog domain: speech/audio is one example; visual images are another. While one may start with the digitized data in which case an analog-domain performance cannot be adequately discussed, there are many other cases where we can discuss the basic characteristics of the original analog data. For example, in audio recordings, we have a fairly good idea on how the frequency characteristics are for recorded signals. Recovering such signals optimally in the sense of analog performance is clearly an important issue.

One of the fundamental problems in this context is that of sample-rate conversion. In commercial applications, there are many different sampling rates employed: for example 48kHz for DAT and 44.1kHz for audio CD. The conversion from one sampling rate to the other becomes necessary. In such a process, it is clearly required that the information loss be as little as possible.

The conventional way of doing this goes as follows: Suppose we want to convert a signal v_d is with sample rate $1/h_1$ Hz to another signal u_d with sample rate $1/h_2$ Hz. Suppose also that there exist (coprime) integers L and M such that $h_1/L = h_2/M$. One first upsamples v_d by factor L, to make the sampling period h_1/L. Suppose that the original signal is perfectly band-limited in the range $|\omega| < 1/2h_1$. One then introduce a digital filter $H_d[z]$ to cut out the undesirable *imaging* component. After this, the obtained signal is downsampled by factor M to become a signal with sampling rate $1/h_2$ Hz.

While this idea is universally employed in the current multirate signal processing, it is based on an artificial assumption: the original signal is perfectly band-limited. This assumption is never precisely satisfied in reality.

Instead of assuming perfect band-limitedness, it is more realistic to assume a prescribed frequency roll-off, and try to optimally reconstruct the original analog signals. We have studied this problem using sampled-data H^∞ control in [3] (The H^∞ criterion in the design of multirate signal processing has been first introduced in [2], which is discussed in the discrete-tiem domain). While this gives rise to a nearly optimal performance, a possible drawback is in its computational efficiency. This is partly due to the framework that we have employed periodically time-varying filters. While this is executable for low conversion ratios, when the conversion involves large integers, the difficulty increases. In the case of CD $\rightarrow$ DAT conversion, where the conversion ratio is $147 : 160$, the resulting filter would become $147 \times 160 = 23,520$ dimensional. This is unrealistic.

To remedy this, we here employ a sequential design procedure. For example, $147 : 160$ can be decomposed into $3 \cdot 7^2 : 2^5 \cdot 5$, and this conversion can be accomplished

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by breaking this into the conversion procedures of 3 : 2, 7 : 5, etc. The overall conversion filter can be obtained by taking the cascade combination of all these. In order that this procedure work, we must require that

- the information loss be minimal at each step, and
- the filter at each step be of low order, so that the order of the product filter is not very high,
- the design procedure for each step be simple.

To satisfy these requirements, we follow the following method:

- we employ an interpolation filter design that optimizes the analog performance;
- we give a decimation filter design that also optimizes the analog performance;
- combining these, we obtain a sample-rate converter design method.

The first has been introduced in [8], and has proven to be very effective when the original signal is not very band-limited. Dually, we introduce an optimal decimation filter design that minimizes analog information loss. This problem is rarely discussed in the literature, but is a crucial component of the present study. When the signal is not quite band-limited, it will be seen to exhibit an even more striking performance difference compared to more conventional digital filter design methods. Finally, we show how they can be combined to give a sample-rate converter, along with a numerical example to exhibit the performance.

The paper is organized as follows: We first introduce the optimal interpolator design problem using upsamplers, following the results of [8]. We then formulate and solve the optimal decimator design problem. These problems are naturally cast into the framework of an infinite-dimensional sampled-data control problem. The infinite-dimensionality stems from the time delay which we allow for signal reconstruction. For this, a fast-sample/fast-hold approximation gives an effective tool for computation. These results are then combined to yield a sample rate converter. The obtained filter is compared with the conventional equiripple design to see the difference, both in time and frequency domain.

2 Problem Formulation

A general construction for a sample-rate converter is shown in Figure 1. For simplicity, we assume L and M are coprime. The first discrete-time signal v_d with sample-rate $1/h_1$ is upsampled by $\uparrow L$:

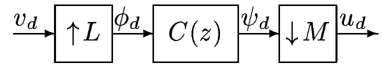

Figure 1: Sample-rate converter

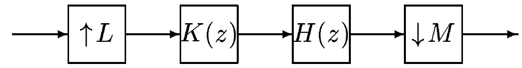

Figure 2: Sample-rate converter

$$\uparrow L : v_d \mapsto \phi_d : \phi_d[k] = \begin{cases} v_d[l], & k = Ll,\ l = 0, 1, \dots \\ 0, & \text{otherwise} \end{cases}$$

by factor L and converted to a higher sample-rate (L/h_1) signal ϕ_d. Then the signal ϕ_d goes through a digital filter $C(z)$ and the filtered signal ψ_d is downsampled by $\downarrow M$:

$$\downarrow M : \psi_d \mapsto u_d : u_d[k] = \psi_d[Mk].$$

by factor M and converted to a lower sample-rate ($1/h_2 = L/Mh_1$) signal u_d. It is possible to formulate a sampled-data design problem, by taking $C(z)$ as an LM-periodic system [3]. A drawback here is that when L and M are large, the filter $C(z)$ would be of very high order (LM-dimensional). This also presents difficulty in numerical computation in executing the filter design.

Another construction for a sample-rate converter is as shown in Figure 2. The idea here is to separete the role of $C(z)$ into two parts: One is the interpolator $K(z)(\uparrow L)$ that interpolates the upsampled signal, and the other is the decimator $(\downarrow M)H(z)$ that decimates the obtained signal to lower the sampling rate to match $1/h_2d$. The advantage is that the filters $K(z)$ and $H(z)$ can be designed separately. Furthermore, if the integers L and M can be decomposed into $L_1L_2\cdots L_m$ and $M_1M_2\cdots M_n$ respectively, the sample-rate converter can be obtained as the cascade composition of respective componenst corresponding these factors as shown in Figure 3.

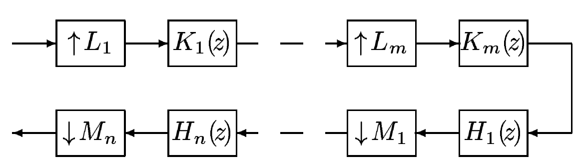

Figure 3: Sample-rate converter

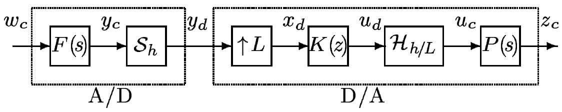

Figure 4: Multirate Signal Reconstruction

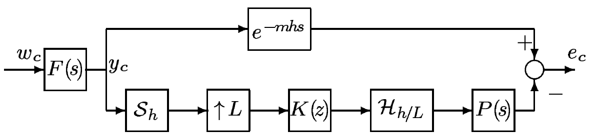

Figure 5: Signal reconstruction error system

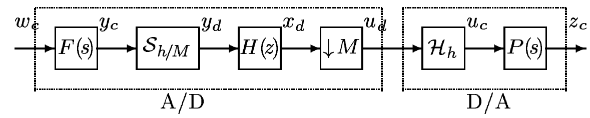

Figure 6: Multirate Signal Reconstruction

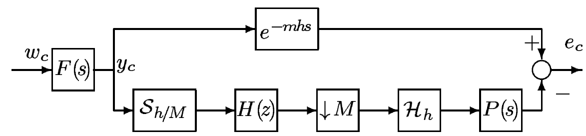

Figure 7: Signal reconstruction error system

2.1 Problem for interpolator design

We start by formulating a design problem for (sub)optimal interpolators. Consider the block diagram Figure 4. The incoming signal w_c first goes through an anti-aliasing filter $F(s)$ and the filtered signal y_c becomes nearly (but not entirely) band-limited. $F(s)$ governs the frequency-domain characteristic of the analog signal y_c. This signal is then sampled by $\mathcal{S}_h$ to become a discrete-time signal y_d with sampling period h.

To restore y_c we usually let it pass through a digital filter, a hold device and then an analog filter. The present setup however places yet one more step: The discrete-time signal y_d is first upsampled by $\uparrow L$, and becomes another discrete-time signal x_d with sampling period h/L. The discrete-time signal x_d is then processed by a digital filter $K(z)$, becomes a continuous-time signal u_c by going through the 0-order hold $\mathcal{H}_{h/L}$ (that works in sampling period h/L), and then becomes the final signal by passing through an analog filter $P(s)$. An advantage here is that one can use a fast hold device $\mathcal{H}_{h/L}$ thereby making more precise signal restoration possible. The objective here is to design the digital filter $K(z)$ for given $F(s)$, L and $P(s)$.

Figure 5 shows the block diagram of the error system for the design. The delay in the upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. Let T_{Iew} denotes the input/output operator from w_c to $e_c := z_c(t) - u_c(t - mh)$. Our design objective is as follows:

Problem 1 *Given stable $F(s)$ and $P(s)$ and an attenuation level $\gamma > 0$, find a digital filter $K(z)$ such that*

$$\|T_{Iew}\| := \sup_{w_c \in L^2[0,\infty)} \frac{\|T_{Iew} w_c\|_2}{\|w_c\|_2} < \gamma. \qquad (1)$$

2.2 Problem for decimator design

We now formulate a design problem for optimal decimators. While this can be considered dually with interpolators, it is less studied in the literature. Downsampling occurs usually in the filter bank design, and its independent design has received less attention.

Consider the block diagram Figure 6. The incoming signal w_c first goes through an anti-aliasing filter $F(s)$ and the filtered signal y_c becomes nearly (but not entirely) band-limited. This signal is then sampled by $\mathcal{S}_{h/M}$ to become a discrete-time signal y_d with sampling period h/M.

The discrete-time signal y_d is first processed by a digital filter $H(z)$. Then the filtered signal x_d is downsampled by $\downarrow M$: and becomes another discrete-time signal u_d with sampling period h. The discrete-time signal u_d is then becomes a continuous-time signal u_c by going through the 0-order hold $\mathcal{H}_h$, and then becomes the final signal by passing through an analog filter $P(s)$. The objective here is to design the digital filter $H(z)$ for given $F(s)$, M and $P(s)$.

Figure 7 shows the block diagram of the error system for the design. The delay in the upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. Let T_{Dew} denotes the input/output operator from w_c to $e_c := z_c(t) - u_c(t - mh)$. Our design objective is as follows:

Problem 2 *Given stable $F(s)$ and $P(s)$ and an attenuation level $\gamma > 0$, find a digital filter $H(z)$ such that*

$$\|T_{Dew}\| := \sup_{w_c \in L^2[0,\infty)} \frac{\|T_{Dew} w_c\|_2}{\|w_c\|_2} < \gamma. \qquad (2)$$

3 Reduction to A Finite-Dimensional Problem

A difficulty in Problem 1 and 2 is that it involves a continuous time-delay, and hence it is an infinite-dimensional problem. Another difficulty is that it contains the upsampler $\uparrow L$ or the downsampler $\downarrow M$, so that it makes the overall system time-varying.

Following the method of [4, 6, 8], however, we can reduce each problem to a finite-dimensional single-rate problem. Let (A_F, B_F, C_F) be a realization of $F(s)$ and define the following operator:

$$\begin{aligned}\mathbf{D}_{11} &: L^2[0,h) \to L^2[0,h)\\ &: w_k \mapsto \int_0^\theta C_F e^{A_F(\theta-\tau)} B_F w_k(\tau) d\tau.\end{aligned}$$

Theorem 1 *1. Suppose $\|\mathbf{D}_{11}\| < \gamma$ for $\gamma > 0$. Then there exist (finite-dimensional) discrete-time systems $G_{I11}(z)$, $G_{I12}(z)$ and $G_{I21}(z)$ such that (1) is equivalent to*

$$\|z^{-m}G_{I11}(z) - G_{I12}(z)\widetilde{K}(z)G_{I21}(z)\|_\infty < \gamma, \tag{3}$$

where $\widetilde{K}(z)$ is the discrete-time lifting of $K(z)$.

2. Suppose $\|\mathbf{D}_{11}\| < \gamma$ for $\gamma > 0$. Then there exist (finite-dimensional) discrete-time systems $G_{D11}(z)$, $G_{D12}(z)$ and $G_{D21}(z)$ such that (2) is equivalent to

$$\|z^{-m}G_{D11}(z) - G_{D12}(z)\widetilde{H}(z)G_{D21}(z)\|_\infty < \gamma, \tag{4}$$

where $\widetilde{H}(z)$ is the discrete-time lifting of $H(z)$.

Proof:

1. We first reduce the problem to a single-rate problem. Define the discrete-time lifting $\mathbf{L}_L$ and its inverse $\mathbf{L}_L^{-1}$ by

$$\begin{aligned}\mathbf{L}_L &:= (\downarrow L)\begin{bmatrix}1\\ z\\ \vdots\\ z^{L-1}\end{bmatrix}\\ \mathbf{L}_L^{-1} &:= \begin{bmatrix}1 & z^{-1} & \cdots & z^{-L+1}\end{bmatrix}(\uparrow L).\end{aligned}$$

Then $K(z)(\uparrow L)$ can be rewritten as

$$\begin{aligned}K(z)(\uparrow L) &= \mathbf{L}_L^{-1}\widetilde{K}(z)\\ \widetilde{K}(z) &:= \mathbf{L}_L K(z)\mathbf{L}_L^{-1}\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}.\end{aligned}$$

$\widetilde{K}(z)$ is an LTI, single-input/L-output system that satisfies

$$K(z) = \begin{bmatrix}1 & z^{-1} & \cdots & z^{-L+1}\end{bmatrix}\widetilde{K}(z^L).$$

Using the generalized hold $\widetilde{\mathcal{H}}_h$ defined by

$$\begin{aligned}&\widetilde{\mathcal{H}}_h : l^2 \ni \mathbf{v} \mapsto u \in L^2, \quad u(kh+\theta) = \mathbf{H}(\theta)\mathbf{v}[k]\\ &\quad \theta \in [0,h), \quad k = 0,1,2,\ldots,\end{aligned}$$

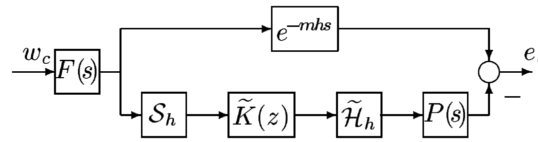

Figure 8: Reduced single-rate problem

where $\mathbf{H}(\theta)$ is the hold function:

$$\mathbf{H}(\theta) := \begin{cases}\begin{bmatrix}1 & 0 & 0 & \ldots & 0\end{bmatrix}, \theta \in [0, h/L)\\ \begin{bmatrix}0 & 1 & 0 & \ldots & 0\end{bmatrix}, \theta \in [h/L, 2h/L)\\ \qquad\cdots\\ \begin{bmatrix}0 & 0 & \ldots & 0 & 1\end{bmatrix}, \theta \in [(L-1)h/L, h)\end{cases}$$

we obtain the identity

$$\mathcal{H}_{h/L}\mathbf{L}_L^{-1} = \widetilde{\mathcal{H}}_h.$$

This yields

$$\mathcal{H}_{h/L}K(z)(\uparrow L)\mathcal{S}_h = \widetilde{\mathcal{H}}_h\widetilde{K}(z)\mathcal{S}_h.$$

Hence Figure 5 is equivalent to Figure 8. We can then invoke the technique of [4] to reduce this to a finite-dimensional design problem (3).

2. Using the discrete-time lifting $\mathbf{L}_M$ we rewrite $(\downarrow M)H(z)$ as

$$\begin{aligned}(\downarrow M)H(z) &= \widetilde{H}(z)\mathbf{L}_M\\ \widetilde{H}(z) &:= \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\mathbf{L}_M H(z)\mathbf{L}_M^{-1}.\end{aligned}$$

$\widetilde{H}(z)$ is an LTI, M-input/single-output system that satisfies

$$H(z) = \widetilde{H}(z^M)\begin{bmatrix}1\\ z\\ \vdots\\ z^{M-1}\end{bmatrix} \tag{5}$$

Using the generalized sampler $\widetilde{\mathcal{S}}_h$ defined by

$$\begin{aligned}&\widetilde{\mathcal{S}}_h : L^2 \ni u \longmapsto \mathbf{v} \in l^2\\ &\mathbf{v}[k] := \begin{bmatrix}u(kh)\\ u(kh+h/M)\\ \vdots\\ u(kh+(M-1)h/M)\end{bmatrix},\\ &\quad k = 0,1,2,\ldots,\end{aligned}$$

we obtain the identity

$$\mathbf{L}_M\mathcal{S}_{h/M} = \widetilde{\mathcal{S}}_h.$$

Hence Figure 7 is equivalent to Figure 9. As has been mentioned above, this can be reduced to a finite-dimentional design problem (3).

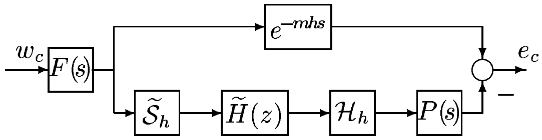

Figure 9: Reduced single-rate problem

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Note that from (5) the filter $H(z)$ may not be causal, thus we adopt the following filter:

$$H(z) = z^{-M}\widetilde{H}(z^M)\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{M-1} \end{bmatrix}$$

4 Approximation via Fast Sample/Hold

While the procedure above reduces Problems 1 and 2 to finite-dimensional H^∞ problems, these are in general not numerically suitable for actual computation; the formulas are quite involved, and not so numerically tractable. It is often more convenient to resort to an approximation method. We employ the fast sample/hold approximation [1, 6]. This method approximates continuouos-time inputs and outputs via a sampler and hold that operate in the period h/L or h/M. The convergence of such an approximation is guaranteed in [7]. Straightforward details and formulas are omitted.

5 A Design Example

We present a design example for the case of changing the sampling period from $h_1 = 1$ to $h_2 = 4/3$. Then we have $L = 3$ and $M = 4$ which are coprime. Let the anti-aliasing filter $F(s)$ and $P(s)$ for the interpolator design be

$$F(s) = \frac{1}{(Ts+1)(0.1Ts+1)}, \quad P(s) = 1,$$

those for the decimator design be

$$F(s) = \frac{1}{(T_2s+1)(0.1T_2s+1)}, \quad P(s) = 1,$$

where $T := 22.05/\pi$, $T_2 := T/L$ which simulate the frequency energy distribution of a typical orchestral music. An approximate design is executed here for $N = L \times 4 = 12$ (interpolator) and $N = M \times 4 = 16$ (decimator). For comparison, we compare it with the equiripple filter[9, 10] of order 31, which is often used in commercial applications.

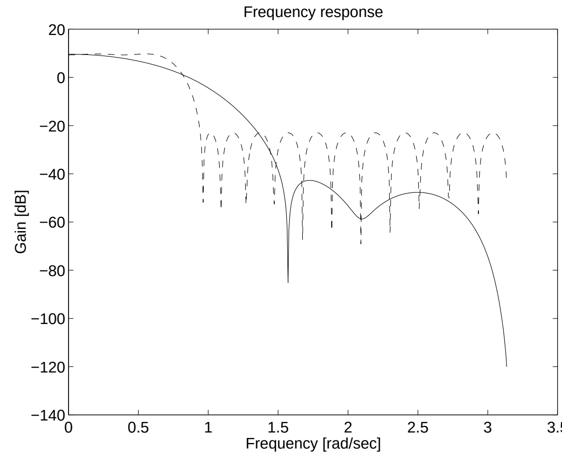

Figure 10: Frequency response of filter: $C(z)$ (solid) and equiripple filter (dash)

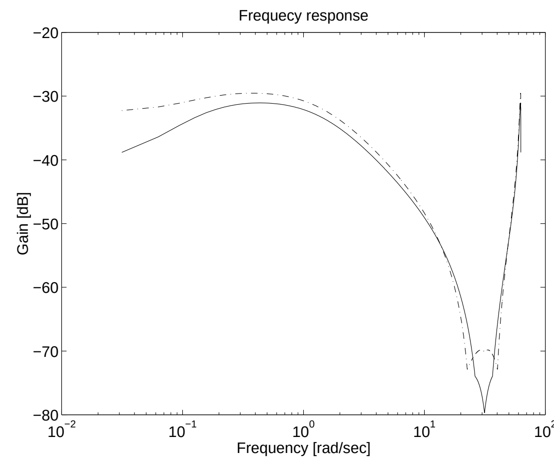

Figure 11: Frequency response of error system T_{ew}: sampled-data H^∞ synthesis (solid) and equiripple filter (dash)

The obtained (sub) optimal interpolation filter $K(z)$ is of order 11 and the decimation filter $H(z)$ of order 15. The sample-rate conversion filter $C(z) = H(z)K(z)$ is of order 22.

Figure 10 shows the gain characteristics of these filters. The equiripple filter shows a sharpest decay beyond the cutoff frequency ($\pi/4$ [rad/sec]) while the sampled-data design shows a rather mild cutoff characteristic.

In spite of these superficial differences, Figure 11 exhibits quite an admirable performance of the sampled-data design.

It is interesting to observe that the slow decay need not yield an inferior design. In fact, due to the underlying analog model (i.e., $F(s)$), there is an important information content beyond the Nyquist frequency, and

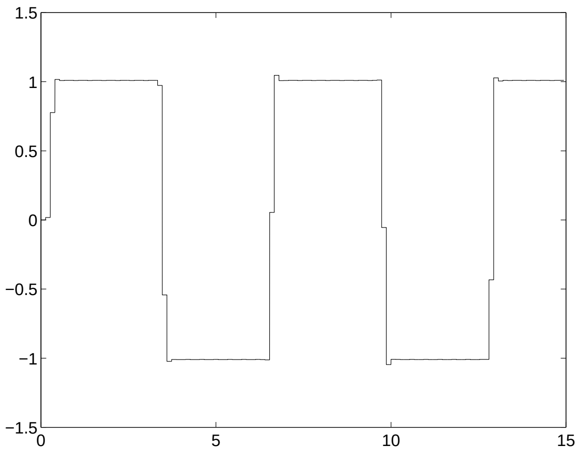

Figure 12: Time response (sampled-data synthesis)

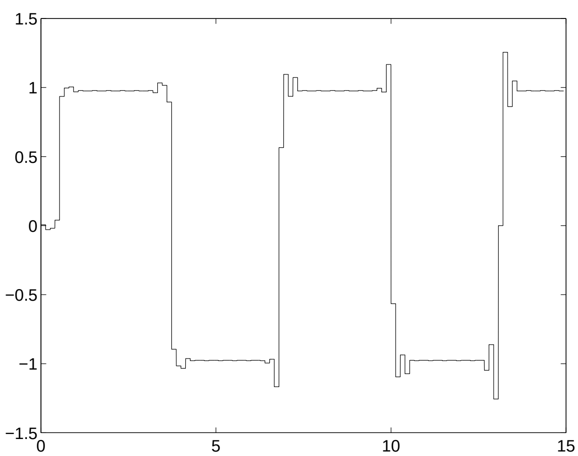

Figure 13: Time response (equiripple filter)

such a slow decay is necessary to retain such information. To see this, let us see the time responses against a rectangular wave in Figures 12, 13:

The equiripple filter shows a large amount of ringing, whereas the one by the sampled-data design has much less peak around the edge. Note also that $C(z)$ is nearly linear phase up to a certain frequency as shown in Figure 14.

6 Concluding Remarks

We have presented a new sampled-data method of designing a digital filter in sample-rate converters. As a result of taking the analog performance into account, the designed converter has yielded much more satisfactory time responses.

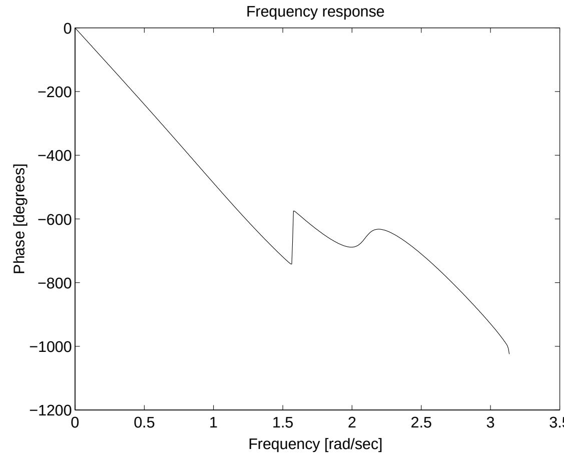

Figure 14: Phase plot of $C(z)$

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