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Optimal FIR Approximation for Discrete-Time IIR Filters

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Abstract

FIR(Finite Impulse Response) filters are often preferred to IIR(Infinite Impulse Response) filters because of their various advantages in stability, phase characteristic possibilities, implementation, etc. This paper proposes a new method to approximate an IIR filter by an FIR filter, which directly yields an optimal approximation with respect to the H^∞ error norm. We formulate this design problem as the minimization of the H^∞ norm of the error transfer function of the difference between the given IIR filter and an FIR filter to be designed. We show that this design problem can be reduced to an LMI (Linear Matrix Inequality). We will also make a comparison via a numerical example with an existing approximation method, known as the Nehari shuffle.

1 Introduction

Digital filters that have finitely many nonzero Markov parameters are called FIR filters (Fig. 1):

$$F(z) = \sum_{k=0}^{M} a_k z^{-k}. \tag{1}$$

They are often preferred to those with infinitely many nonzero coefficients known as IIR filters (Fig. 2)

$$F(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1 + \sum_{k=1}^{N} b_k z^{-k}} \tag{2}$$

for the following reasons [7]:

- FIR filters are intrinsically stable; the stability issue is a non-issue.

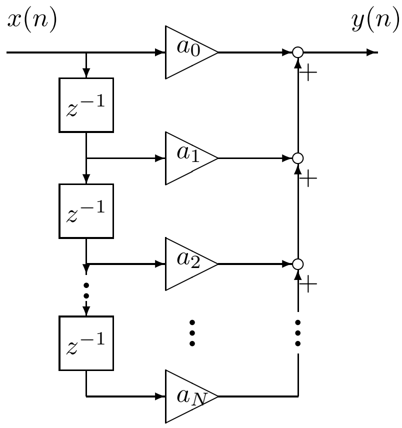

Figure 1: FIR filter

- They can easily realize various features that are not possible or are difficult to achieve with IIR filters, e.g., linear phase property.

- They can be free from certain problems in implementation, for example, limit cycles, attributed to quantization and the existence of a feedback loop in IIR filters.

On the other hand, a design process may have to start with an IIR filter for variety of reasons. For example, we have a large number of continuous-time filters available, and a digital filter may be obtained by discretizing one of them. It is then desired that such an IIR filter be approximated by an FIR filter. An easy, but not necessarily optimal, way of doing this is to just truncate the Markov parameters of the IIR filter at a desired number of steps. This may however lead to either a very high-dimensional filter with good approximation, or a lower dimensional filter with an unsatisfactory approximation, depending on the truncation point.

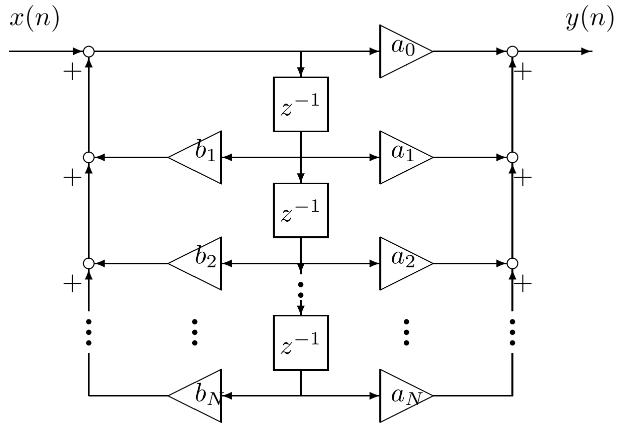

Figure 2: IIR filter

The following problem is thus very natural and of importance:

Problem 1 Given an IIR filter $K(z)$ and a positive integer N, find an optimal FIR approximant $K_f(z)$ that has order N and approximates $K(z)$ with respect to a certain performance measure.

There is a very elegant method called the Nehari shuffle, proposed by Kootsookos et al. [3, 4]. Its basic idea may be described as follows: For a given IIR filter G and a desired degree $r-1$ that an approximating FIR filter should assume, one first truncates the impulse response of G to the first r steps. This is a mere truncation, and it may induce a large error. One then takes its residual G_1, and approximates G_1 by another FIR filter of length r by using a Nehari approximation of a shifted and mirror image version of G_1 followed by truncation. This truncation produces another residual; the process can be continued, and the approximation can be improved in each step. (Details may be found in [5]. An advantage here is that this procedure gives rise to certain a priori and a posteriori error bounds. On the other hand, it does not necessarily give an optimal approximation with respect to the H^∞ norm.

In contrast to the Nehari shuffle, we here propose a method that directly deals with (sub)optimal approximants with respect to a (weighted) H^∞ error norm. It is shown that

- the design problem is reducible to an LMI [1]; and
- the obtained filter can be made close to be optimal by an iterative procedure.

A comparison with the Nehari shuffle is made for the Chebyshev filter of order 8, which has been studied in detail in [4].

2 FIR Approximation Problem

Consider the block diagram Fig. 3.

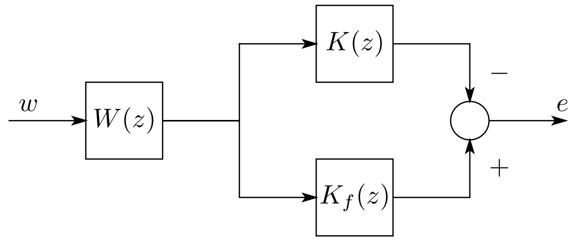

Figure 3: Error system

$K(z)$ is a given (rational and stable) IIR filter, $W(z)$ is a proper and rational weighting function, and $K_f(z)$ is an FIR filter of order N. The objective here is to find $K_f(z)$ that makes the H^∞ error norm less than a prespecified bound γ.

Introduce state space realizations

$$\begin{aligned} W(z): &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \\ K(z): &= \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right], \end{aligned}$$

and

$$\begin{aligned} K_f(z) &= \sum_{k=0}^{N} a_k z^{-k} = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f(\alpha) & D_f(\alpha) \end{array}\right], \\ C_f(\alpha) &= \left[\begin{array}{cccc} a_N & a_{N-1} & \dots & a_1 \end{array}\right], \\ D_f(\alpha) &= a_0, \\ \alpha &= \left[\begin{array}{cccc} a_N & a_{N-1} & \dots & a_0 \end{array}\right]. \end{aligned}$$

where $\alpha = \left[\begin{array}{cccc} a_N & a_{N-1} & \dots & a_0 \end{array}\right]$ denote the Markov parameters of the filter $K_f(z)$ to be designed. The matrices A_f and B_f are defined as follows:

$$\begin{aligned} A_f &= \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{array}\right], \\ B_f &= \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array}\right], \end{aligned}$$

and they contain just zeros and ones.

Denote by $T_{ew}(z)$ the transfer function from w to e in Fig. 3. Its realization is then given as follows:

$$T_{ew}(z) =: \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W(\alpha) & D_W(\alpha) \end{array}\right],$$

$$A_W = \begin{bmatrix} A & 0 & 0 \\ B_k C & A_k & 0 \\ B_f C & 0 & A_f \end{bmatrix},$$

$$B_W = \begin{bmatrix} B \\ B_k D \\ B_f D \end{bmatrix},$$

$$C_W = \begin{bmatrix} (D_f(\alpha) - D_k)C & -C_k & C_f(\alpha) \end{bmatrix},$$

$$D_W = \begin{bmatrix} (D_k + D_f(\alpha))D \end{bmatrix}.$$

The important asset here is that the design parameter α appears only in the C and D matrices linearly, and the underlying structure is of the one-block type. Hence the overall transfer operator is linear in α, and the design problem of choosing α to minimize the H_∞ norm can be expected to become a linear matrix inequality. In fact, the bounded real lemma [1] readily yields the following:

Theorem 1 $\|T_{ew}\| < \gamma$ if and only if there exists $P > 0$ such that

$$\begin{bmatrix} A_W^T P A_W - P & A_W^T P B_W & C_W^T \\ B_W^T P A_W & -\gamma I + B_W^T P B_W & D_W^T \\ C_W & D_W & -\gamma I \end{bmatrix} < 0. \tag{3}$$

Proof: By the bounded real lemma [1], $\|T_{ew}\| < \gamma$ is equivalent to the condition that there exists a matrix $\tilde{P} > 0$ such that

$$Q^T \begin{bmatrix} \tilde{P} & 0 \\ 0 & I \end{bmatrix} Q < \begin{bmatrix} \tilde{P} & 0 \\ 0 & \gamma^2 I \end{bmatrix}, \tag{4}$$

where

$$Q =: \begin{bmatrix} A_W & B_W \\ C_W(\alpha) & D_W(\alpha) \end{bmatrix}.$$

Although the inequality (4) is not affine in α, it can be converted to an affine one by the Schur complement [1]:

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0$$

is equivalent to $\Phi_{22} < 0$ and $\Phi_{11} < \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^T$. By dividing the inequality (4) by $\gamma > 0$, we obtain

$$\begin{bmatrix} A_W^T \gamma^{-1}\tilde{P} A_W - \gamma^{-1}\tilde{P} & A_W^T \gamma^{-1}\tilde{P} B_W \\ B_W^T \gamma^{-1}\tilde{P} A_W & B_W^T \gamma^{-1}\tilde{P} B_W - \gamma I \end{bmatrix}$$
$$< \begin{bmatrix} C_W^T \\ D_W^T \end{bmatrix} (-\gamma^{-1} I) \begin{bmatrix} C_W & D_W \end{bmatrix}.$$

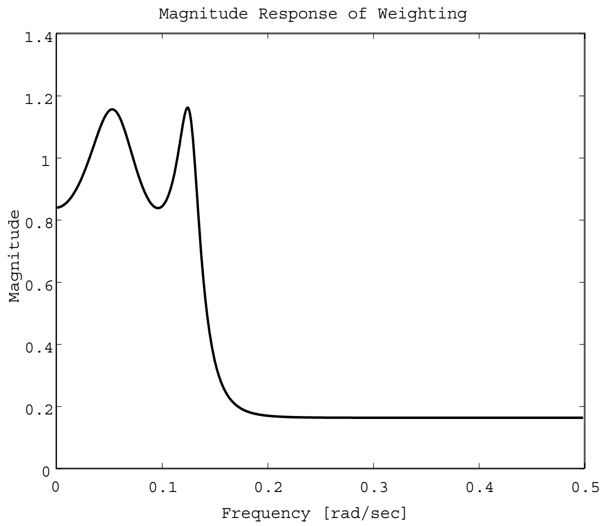

Figure 4: Inverse of the weighting function

Then by using the Sure complement for

$$\Phi_{11} := \begin{bmatrix} A_W^T P A_W - P & A_W^T P B_W \\ B_W^T P A_W & B_W^T P B_W - \gamma I \end{bmatrix}^T,$$

$$\Phi_{22} := -\gamma I,$$

$$\Phi_{12} := \begin{bmatrix} C_W & D_W \end{bmatrix}^T,$$

where $P := \gamma^{-1}\tilde{P} > 0$, we obtain the inequality (3). ■

The obtained condition is an LMI in α, and can be effectively solved by standard MATLAB routines [2].

3 Numerical Example

3.1 Comparison of H^∞ Design via LMI and the Nehari Shuffle

Take the following Chebyshev filter of order 8

$$K(z) = 10^{-3} \times \frac{0.04705z^8 + 0.3764z^7 + 1.317z^6}{z^8 - 4.953z^7 + 11.71z^6}$$
$$\frac{+2.635z^5 + 3.294z^4 + 2.635z^3 + 1.317z^2}{-16.95z^5 + 16.29z^4 - 10.58z^3 + 4.552z^2}$$
$$\frac{+0.3764z + 0.04705}{-1.161z + 0.1369}.$$

as a target filter to be approximated. This has been studied thoroughly by Kootsookos and Bitmead [4] using the Nehari shuffle, and is suitable for comparison with the present method. For simplicity, we confine ourselves to approximations by FIR filters with 32 tap coefficients (of order 31).

The design depends crucially on the choice of the weight $W(z)$. One natural choice ([5]) would be to take $W(z)$ to be equal to $K^{-1}(z)$ (or some variant of it having the same gain on the unit circle, since K is not minimum phase). This is relative error approximation, where (approximately) dB and phase errors are weighted uniformly with frequency. Since the error criterion in Fig. 3 is taken with respect to the H^∞ norm, it approximates equal amplitude at all frequencies, and this will have the effect of attenuating the stop-band error with the weight of $K^{-1}(z)$ (which is very large) while maintaining reasonable pass-band characteristic. Unfortunately, however, due to the very small gain of $K(z)$ in the stop-band, this will make the solution of the approximation problem Fig. 3 numerically hard. Neither the Nehari shuffle nor the LMI method gave a satisfactory result in this case. Hence one should sacrifice some stop-band attenuation to obtain a reasonable $W(z)$. There is also a trade-off, empirically observed, between the stop-band attenuation and the pass-band ripples.

Kootsookos and Bitmead [4] thus employed the weight as depicted in Fig. 4. To be precise, the frequency response shown here is the inverse of the para-Hermitian conjugate of the weight function. The reason for taking the para-Hermitian conjugate is that the Nehari shuffle makes use of causal approximations of anti-causal transfer functions, so that we must us reciprocals of the poles and zeros. Then by taking the inverse, the weight attenuates the stop-band by the inverse of its gain and approximately shapes the pass-band as it is in the pass band. On the other hand, for the FIR approximation as in Fig. 3, we simply take the inverse of this weight, since we do not need to make the weight anti-stable.

The gain responses of the obtained FIR filters based on the Nehari shuffle and Theorem 1 are given in Fig. 5. Fig. 6 shows their phase plots.

We see that the gain of the H^∞ approximant shows smaller pass-band ripples and better stop-band attenuation than for the Nehari shuffle. The phase characteristics are about the same up to the edge of the transition band.

Fig. 7 shows the error magnitude response. The design by the LMI method has the advantage of 5–7 dB over the one by the Nehari shuffle.

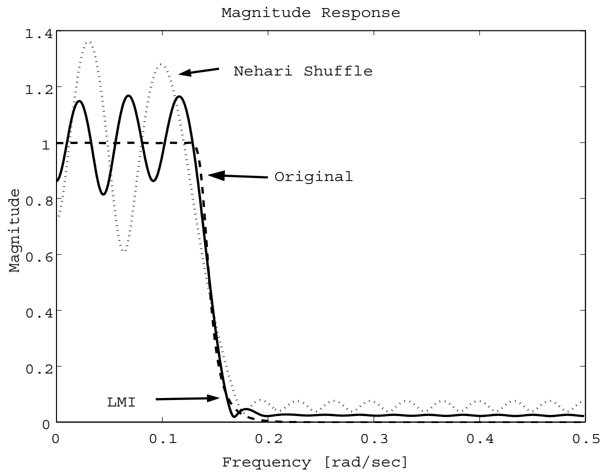

Figure 5: Gain responses of FIR approximants with weight function in Fig. 4: H^∞ via LMI (solid); Nehari shuffle (dots); original IIR Chebyshev filter (- · -)

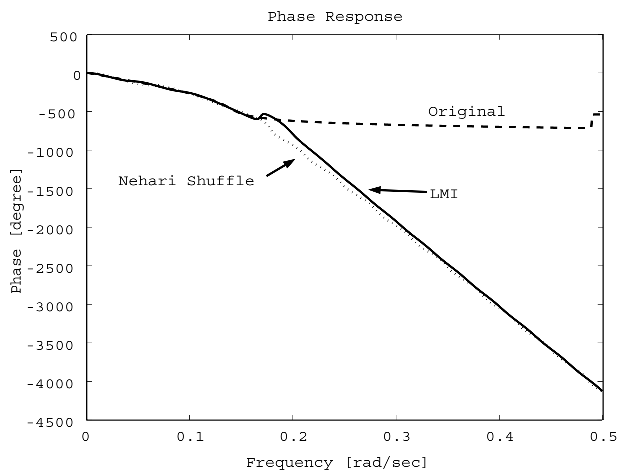

Figure 6: Phase plots of FIR approximants with weight function in Fig. 4: H^∞ via LMI (solid); Nehari shuffle (dots); Chebyshev (- · -)

3.2 Trade-off Between Pass-Band and Stop-Band Characteristics

The design in the previous subsection depends crucially on the weighting function. It may be considered desirable to obtain smaller pass-band ripples while maintaining reasonable stop-band attenuation. In this section we attempt to see how the choice of a weighting function affects the overall approximation.

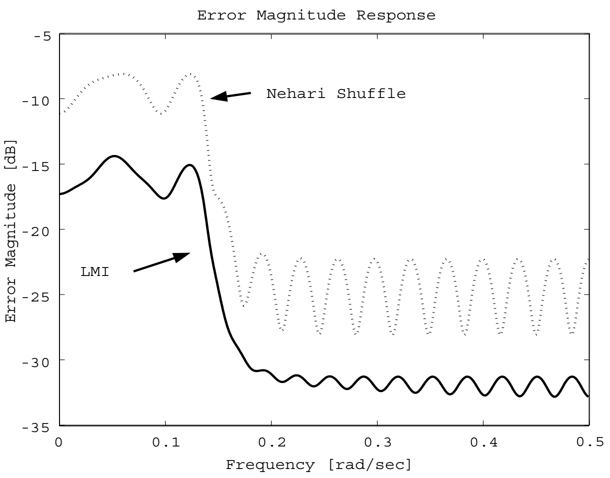

Figure 7: Gain of the error $K - K_f$: H^∞ design via LMI (solid); Nehari shuffle (dots)

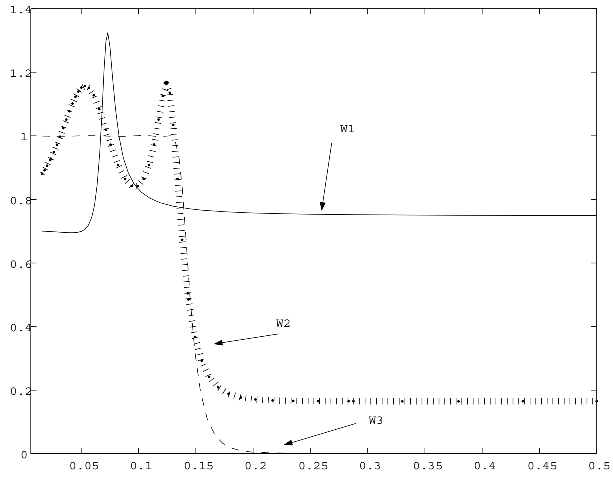

Figure 8: Inverse of the gain of the weighting functions

We consider the following three weighting functions:

$$\begin{aligned} W_1 &= \frac{0.7661z^2 - 1.305z + 0.675}{z^2 - 1.735z + 0.9289}, \\ W_2 &= \frac{0.2831z^4 - 0.5515z^3 + 0.5416z^2}{z^4 - 2.865z^3 + 3.6z^2} \\ &\quad \frac{-0.2708z + 0.05882}{-2.268z + 0.6056}, \\ W_3 &= \frac{0.01444z^7 - 0.007838z^6 + 0.01902z^5}{z^7 - 4.229z^6 + 8.561z^5} \\ &\quad \frac{-0.004448 + 0.006697z^3 - 0.0001857z^2}{-10.43z^4 + 8.172z^3 - 4.089z^2} \\ &\quad \frac{+0.0005287z + 1.134e^{-5}}{+1.206z - 0.1613}. \end{aligned}$$

These functions W_1, W_2, W_3 are, respectively, obtained as the 2nd, 4th, 7th-order Hankel norm approximations [5] of the IIR Chebyshev filter to be approximated. The weight W_2 is the same as that used in the previous section. Their magnitude frequency responses are shown in Fig. 8.

Fig. 9 and Fig. 10 show the resulting gain and phase responses of the FIR filters designed with the respective weighting functions. Fig. 9 in particular shows that there is a clear trade-off between the magnitude of the pass-band ripples and the stop-band attenuation. That is, if we attempt to decrease the stop-band error, we must sacrifice the pass-band characteristic (i.e., accept larger ripples), and vice versa.

Fig. 11 shows the error magnitude responses. Table 1 also shows the H^∞ and H^2 error norms. Interestingly, the design via W_1 exhibits the best overall approximation in both performance measures, although its stop-band attenuation is not as good as those obtained using W_2 and W_3. Note also that the design resulting from W_3 approximates the phase characteristic of the original filter up to the edge of the stop-band as Fig. 10 shows.

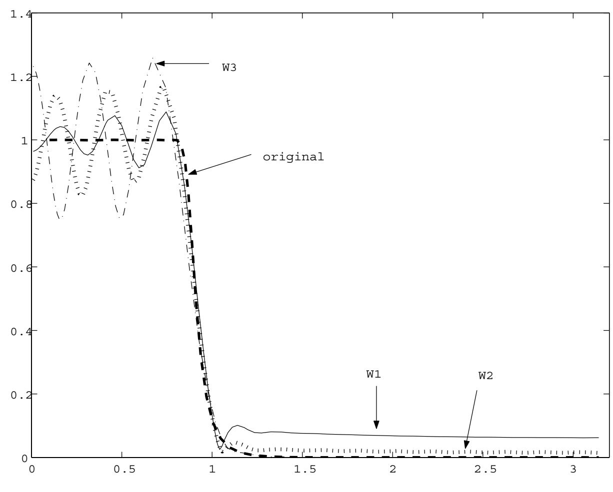

Figure 9: Gain responses of FIR filters via LMI

4 Conclusion

We have given an LMI solution to the optimal H^∞ approximation of IIR filters via FIR filters. A comparison with the Nehari shuffle is made with a numerical example, and it is observed that the LMI solution generally performs better. Another numerical study also indicates that there is a trade-off between the pass-band and stop-band approximation characteristics.

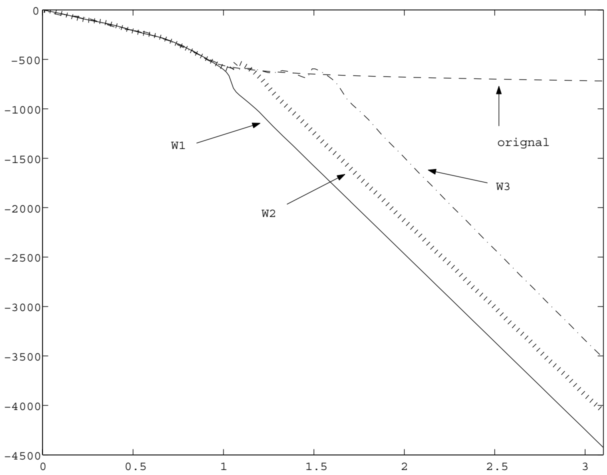

Figure 10: Phase responses of FIR filters via LMI

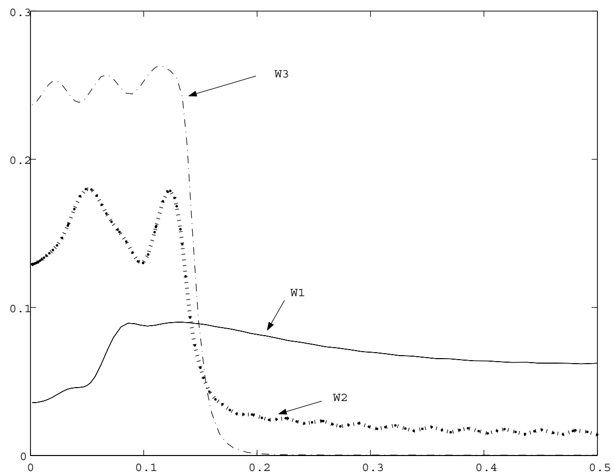

Figure 11: Gain responses of the error $K_f - K$

Table 1: H^∞ and H^2 error norm

	W_1	W_2	W_3
H^∞ error norm	0.0954	0.1838	0.2627
H^2 error norm	0.0713	0.0840	0.1333

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