

Maximum Hands-Off Control: A Paradigm of Control Effort Minimization

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Abstract—In this paper, we propose a paradigm of control, called a maximum hands-off control. A hands-off control is defined as a control that has a short support per unit time. The maximum hands-off control is the minimum support (or sparsest) per unit time among all controls that achieve control objectives. For finite horizon continuous-time control, we show the equivalence between the maximum hands-off control and L^1 -optimal control under a uniqueness assumption called normality. This result rationalizes the use of L^1 optimality in computing a maximum hands-off control. The same result is obtained for discrete-time hands-off control. We also propose an L^1/L^2 -optimal control to obtain a smooth hands-off control. Furthermore, we give a self-triggered feedback control algorithm for linear time-invariant systems, which achieves a given sparsity rate and practical stability in the case of plant disturbances. An example is included to illustrate the effectiveness of the proposed control.

Index Terms—Hands-off control, sparsity, L^1 -optimal control, self-triggered control, stability, nonlinear systems

I. INTRODUCTION

In practical control systems, we often need to minimize the control effort so as to achieve control objectives under limitations in equipment such as actuators, sensors, and networks. For example, the energy (or L^2 -norm) of a control signal can be minimized to prevent engine overheating or to reduce transmission cost by means of a standard LQ (linear quadratic) control problem; see e.g., [1]. Another example is the *minimum fuel* control, discussed in e.g., [2], [3], in which the total expenditure of fuel is minimized with the L^1 norm of the control.

Alternatively, in some situations, the control effort can be dramatically reduced by holding the control value *exactly zero* over a time interval. We call such control a *hands-off control*. A motivation for hands-off control is a stop-start system in automobiles. It is a hands-off control; it automatically shuts down the engine to avoid it idling for long periods of time. By this, we can reduce CO or CO₂ emissions as well as fuel consumption [14]. This strategy is also used in

electric/hybrid vehicles [8]; the internal combustion engine is stopped when the vehicle is at a stop or the speed is lower than a preset threshold, and the electric motor is alternatively used. Thus hands-off control also has potential for solving environmental problems. In railway vehicles, hands-off control, called *coasting*, is used to reduce energy consumption [34]. Furthermore, hands-off control is desirable for networked and embedded systems since the communication channel is not used during a period of zero-valued control. This property is advantageous in particular for wireless communications [28], [32] and networked control systems [36], [26], [38], [31]. Motivated by these applications, we propose a paradigm of control, called *maximum hands-off control* that maximizes the time interval over which the control is exactly zero.

The hands-off property is related to *sparsity*, or the L^0 “norm” (the quotation marks indicate that this is not a norm; see Section II below) of a signal, defined by the total length of the intervals over which the signal takes non-zero values. The maximum hands-off control, in other words, seeks the *sparsest* (or L^0 -optimal) control among all admissible controls. The notion of sparsity has been recently adapted to control systems, including works on model predictive control [36], [19], [22], [39], [38], system gain analysis [41], sparse controller design [17], state estimation [9], to name a few. The maximum hands-off control is also related to the minimum attention control [5], and also to the approach by Donkers et al. [12], which maximizes the time between consecutive execution of the control tasks. The minimum attention control minimizes the number of switching per unit time. In contrast, the maximum hands-off control does not necessarily minimize the number of switching, although we show this number is bounded for linear systems.

The maximum hands-off control (or L^0 -optimal control) problem is hard to solve since the cost function is non-convex and discontinuous.¹ To overcome the difficulty, one can adopt L^1 optimality as a convex relaxation of the problem, as often used in *compressed sensing* [13], [6]. Compressed sensing has shown by theory and experiments that sparse high-dimensional signals can be reconstructed from incomplete measurements by using ℓ^1 optimization; see e.g., [15], [16], [23] for details.

Interestingly, a finite horizon L^1 -optimal (or minimum fuel) control has been known to have such a sparsity property, traditionally called “*bang-off-bang*” [3]. Based on this, L^1 -optimal

¹A preliminary version of parts of this work was presented in [37].

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¹Very recently, L^p control with $p \in [0, 1)$ has been investigated in [27], which introduces regularization terms to guarantee the existence of optimal solutions.

control has been recently investigated for designing sparse control [33], [7], [29]. Although advantage has implicitly been taken of the sparsity property for minimizing the L^1 norm, we are not aware of results on the theoretical connection between sparsity and L^1 optimality of the control. In the present manuscript, we prove that a solution to an L^1 -optimal control problem gives a maximum hands-off control, and vice versa. As a result, the sparsest solution (i.e., the maximum hands-off control) can be obtained by solving an L^1 -optimal control problem. The same result is obtained for discrete-time hands-off control. We also propose L^1/L^2 -optimal control to avoid the discontinuous property of "bang-off-bang" in maximum hands-off control. We show that the L^1/L^2 -optimal control is an intermediate control between the maximum hands-off (or L^1 -optimal) control and the minimum energy (or L^2 -optimal) control, in the sense that the L^1 and L^2 controls are the limiting instances of the L^1/L^2 -optimal control.

We also extend the maximum hands-off control to feedback control for linear time-invariant, reachable, and nonsingular systems by a *self-triggering* approach [42], [35], [24], [4]. For this, we define sparsity of infinite horizon control signals by the *sparsity rate*, the L^0 norm per unit time. We give a self-triggered feedback control algorithm that achieves a given sparsity rate and practical stability in the presence of plant disturbances. Simulations studies demonstrate the effectiveness of the proposed control method.

The present manuscript expands upon our recent conference contribution [37] by incorporating feedback control into the formulation.

The remainder of this article is organized as follows: In Section II, we give mathematical preliminaries for our subsequent discussion. In Section III, we formulate the maximum hands-off control problem. Section IV is the main part of this paper, in which we introduce L^1 -optimal control as relaxation of the maximum hands-off control, and establish the theoretical connection between them. We also analyze discrete-time hands-off control in this section. In Section V, we propose L^1/L^2 -optimal control for a smooth hands-off control in this section. In Section VI, we address the feedback hands-off control. Section VII presents control design examples to illustrate the effectiveness of our method. In Section VIII, we offer concluding remarks.

II. MATHEMATICAL PRELIMINARIES

For a vector $\mathbf{x} \in \mathbb{R}^n$, we define its norm by

$$\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^\top \mathbf{x}},$$

and for a matrix $A \in \mathbb{R}^{n \times n}$,

$$\|A\| \triangleq \max_{\mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

For a continuous-time signal $u(t)$ over a time interval $[0, T]$, we define its L^p norm with $p \in [1, \infty)$ by

$$\|u\|_p \triangleq \left(\int_0^T |u(t)|^p dt \right)^{1/p}, \quad (1)$$

and let $L^p[0, T]$ consist of all u for which $\|u\|_p < \infty$. Note that we can also define (1) for $p \in (0, 1)$, which is not a

norm (It fails to satisfy the triangle inequality.). We define the support set of u , denoted by $\text{supp}(u)$, the closure of the set

$$\{t \in [0, T] : u(t) \neq 0\}.$$

Then we define the L^0 "norm" of measurable function u as the length of its support, that is,

$$\|u\|_0 \triangleq m_L(\text{supp}(u)),$$

where m_L is the Lebesgue measure on \mathbb{R} . Note that the L^0 "norm" is not a norm since it fails to satisfy the positive homogeneity property, that is, for any non-zero scalar α such that $|\alpha| \neq 1$, we have

$$\|\alpha u\|_0 = \|u\|_0 \neq |\alpha| \|u\|_0, \quad \forall u \neq 0.$$

The notation $\|\cdot\|_0$ may be however justified from the fact that if $u \in L^1[0, T]$, then $\|u\|_p < \infty$ for any $p \in (0, 1)$ and

$$\lim_{p \rightarrow 0} \|u\|_p^p = \|u\|_0,$$

which can be proved by using Lebesgue's monotone convergence theorem [40]. For more details of L^p when $p \in [0, 1)$, see [30]. For a function $\mathbf{f} = [f_1, \dots, f_n]^\top : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the Jacobian \mathbf{f}' is defined by

$$\mathbf{f}'(\mathbf{x}) \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix},$$

where $\mathbf{x} = [x_1, \dots, x_n]^\top$. For functions f and g , we denote by $f \circ g$ the composite function $f(g(\cdot))$.

III. MAXIMUM HANDS-OFF CONTROL PROBLEM

In this section, we formulate the maximum hands-off control problem. We first define the *sparsity rate*, the L^0 norm of a signal per unit time, of finite-horizon continuous-time signals.

Definition 1 (Sparsity rate): For measurable function u on $[0, T]$, $T > 0$, the *sparsity rate* is defined by

$$R_T(u) := \frac{1}{T} \|u\|_0. \quad (2)$$

Note that for any measurable u , $0 \leq R_T(u) \leq 1$. If $R_T(u) \ll 1$, we say u is *sparse*.² The control objective is, roughly speaking, to design a control u which is as sparse as possible, whilst satisfying performance criteria. For that purpose, we will first focus on finite T and then, in Section VI, study the infinite horizon case, where $T \rightarrow \infty$.

To formulate the control problem, we consider nonlinear multi-input plant models of the form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x}(t)) u_i(t), \quad t \in [0, T], \quad (3)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, u_1, \dots, u_m are the scalar control inputs, \mathbf{f} and \mathbf{g}_i are functions on \mathbb{R}^n . We assume that $\mathbf{f}(\mathbf{x})$, $\mathbf{g}_i(\mathbf{x})$, and their Jacobians $\mathbf{f}'(\mathbf{x})$, $\mathbf{g}'_i(\mathbf{x})$ are continuous. We use the vector representation $\mathbf{u} \triangleq [u_1, \dots, u_m]^\top$.

²This is analogous to the sparsity of a vector. When a vector has a small number of non-zero elements relative to the vector size, then it is called *sparse*. See [15], [16], [23] for details.

The control $\{\mathbf{u}(t) : t \in [0, T]\}$ is chosen to drive the state $\mathbf{x}(t)$ from a given initial state

$$\mathbf{x}(0) = \boldsymbol{\xi}, \quad (4)$$

to the origin at a fixed final time $T > 0$, that is,

$$\mathbf{x}(T) = \mathbf{0}. \quad (5)$$

Also, the components of the control $\mathbf{u}(t)$ are constrained in magnitude by

$$\max_i |u_i(t)| \leq 1, \quad (6)$$

for all $t \in [0, T]$. We call a control $\{\mathbf{u}(t) : t \in [0, T]\} \in L^1[0, T]$ *admissible* if it satisfies (6) for all $t \in [0, T]$, and the resultant state $\mathbf{x}(t)$ from (3) satisfies boundary conditions (4) and (5). We denote by $\mathcal{U}(T, \boldsymbol{\xi})$ the set of all admissible controls.

To consider control in $\mathcal{U}(T, \boldsymbol{\xi})$, it is necessary that $\mathcal{U}(T, \boldsymbol{\xi})$ is non empty. This property is basically related to the *minimum-time control* formulated as follows:

Problem 2 (Minimum-time control): Find a control $\mathbf{u} \in L^1[0, T]$ that satisfies (6), and drives \mathbf{x} from initial state $\boldsymbol{\xi} \in \mathbb{R}^n$ to the origin $\mathbf{0}$ in minimum time. ■

Let $T^*(\boldsymbol{\xi})$ denote the minimum time (or the value function) of Problem 2. Also, we define the reachable set as follows:³

Definition 3 (Reachable set): We define the reachable set at time $t \in [0, \infty)$ by

$$\mathcal{R}(t) \triangleq \{\boldsymbol{\xi} \in \mathbb{R}^n : T^*(\boldsymbol{\xi}) \leq t\}. \quad (7)$$

and the reachability set

$$\mathcal{R} \triangleq \bigcup_{t \geq 0} \mathcal{R}(t). \quad (8)$$

To guarantee that $\mathcal{U}(T, \boldsymbol{\xi})$ is non-empty, we introduce the standing assumptions:

- 1) $\boldsymbol{\xi} \in \mathcal{R}$,
- 2) $T > T^*(\boldsymbol{\xi})$.

Now let us formulate our control problem. The *maximum hands-off control* is a control that is the *sparsest* among all admissible controls in $\mathcal{U}(T, \boldsymbol{\xi})$. In other words, we try to find a control that maximizes the time interval over which the control $\mathbf{u}(t)$ is exactly zero.⁴ We state the associated optimal control problem as follows:

Problem 4 (Maximum hands-off control): Find an admissible control on $[0, T]$, $\mathbf{u} \in \mathcal{U}(T, \boldsymbol{\xi})$, that minimizes the sum of sparsity rates:

$$J_0(\mathbf{u}) \triangleq \sum_{i=1}^m \lambda_i R_T(u_i) = \frac{1}{T} \sum_{i=1}^m \lambda_i \|u_i\|_0, \quad (9)$$

where $\lambda_1 > 0, \dots, \lambda_m > 0$ are given weights. ■

This control problem is quite difficult to solve since the objective function is highly nonlinear and non-smooth. In the next section, we discuss convex relaxation of the maximum

³For linear systems, the reachable set is known to have nice properties such as convexity and compactness [25], [20].

⁴More precisely, the maximum hands-off control minimizes the Lebesgue measure of the support. Hence, the values on the sets of measure zero are ignored and treated as zero in this setup.

hands-off control problem, which gives the exact solution of Problem 4 under some assumptions.

Remark 5: The input constraint (6) is necessary. Let us consider the integrator $\dot{x}(t) = u(t)$ and remove the constraint (6). Then for any $\epsilon > 0$, the following control is an admissible control

$$u_\epsilon(t) = \begin{cases} \xi/\epsilon, & t \in [0, \epsilon), \\ 0, & t \in [\epsilon, T], \end{cases}$$

which has arbitrarily small L^0 norm. But $\lim_{\epsilon \rightarrow 0} u_\epsilon$ is not a function, so called Dirac's delta, and hence is not in L^1 . In this case, the maximum hands-off problem has no solution.

IV. SOLUTION TO MAXIMUM HANDS-OFF CONTROL PROBLEM

In this section we will show how the maximum hands-off control can be solved in closed form.

A. Convex Relaxation

Here we consider convex relaxation of the maximum hands-off control problem. We replace $\|u_i\|_0$ in (9) with L^1 norm $\|u_i\|_1$, and obtain the following L^1 -optimal control problem, also known as *minimum fuel control* discussed in e.g. [2], [3].

Problem 6 (L^1 -optimal control): Find an admissible control $\mathbf{u} \in \mathcal{U}(T, \boldsymbol{\xi})$ on $[0, T]$ that minimizes

$$J_1(\mathbf{u}) \triangleq \frac{1}{T} \sum_{i=1}^m \lambda_i \|u_i\|_1 = \frac{1}{T} \sum_{i=1}^m \lambda_i \int_0^T |u_i(t)| dt, \quad (10)$$

where $\lambda_1 > 0, \dots, \lambda_m > 0$ are given weights. ■

The objective function (10) is convex in \mathbf{u} and this control problem is much easier to solve than the maximum hands-off control problem (Problem 4). The main contribution of this section is that we prove the solution set of Problem 6 is equivalent to that of Problem 4, under the assumption of normality. Before proving this property, we review L^1 -optimal control in the next subsection.

B. Review of L^1 -Optimal Control

Here we briefly review the L^1 -optimal control problem (Problem 6) based on the discussion in [3, Section 6-13].

Let us first form the Hamiltonian function for the L^1 -optimal control problem as

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}) = \frac{1}{T} \sum_{i=1}^m \lambda_i |u_i| + \mathbf{p}^\top \left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x}) u_i \right), \quad (11)$$

where \mathbf{p} is the costate (or adjoint) vector [3, Section 5-7]. Assume that $\mathbf{u}^* = [u_1^*, \dots, u_m^*]^\top$ is an L^1 -optimal control and \mathbf{x}^* is the resultant state trajectory. According to the minimum principle, there exists a costate \mathbf{p}^* such that the optimal control \mathbf{u}^* satisfies

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}^*(t)) \leq H(\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}(t)),$$

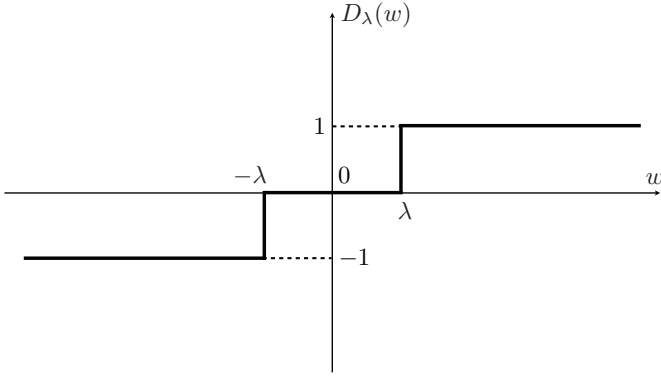


Fig. 1. Dead-zone function $D_\lambda(w)$

for for all $t \in [0, T]$ and all admissible \mathbf{u} . The optimal state \mathbf{x}^* and costate \mathbf{p}^* satisfies the canonical equations

$$\begin{aligned} \frac{d\mathbf{x}^*(t)}{dt} &= \mathbf{f}(\mathbf{x}^*(t)) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x}^*(t)) u_i^*(t), \\ \frac{d\mathbf{p}^*(t)}{dt} &= -\mathbf{f}'(\mathbf{x}^*(t))^\top \mathbf{p}^*(t) \\ &\quad - \sum_{i=1}^m u_i^*(t) \mathbf{g}'_i(\mathbf{x}^*(t))^\top \mathbf{p}^*(t), \end{aligned}$$

with boundary conditions

$$\mathbf{x}^*(0) = \boldsymbol{\xi}, \quad \mathbf{x}^*(T) = \mathbf{0}.$$

The minimizer $\mathbf{u}^* = [u_1^*, \dots, u_m^*]^\top$ of the Hamiltonian in (11) is given by

$$u_i^*(t) = -D_{\lambda_i/T}(\mathbf{g}_i(\mathbf{x}^*(t))^\top \mathbf{p}^*(t)), \quad t \in [0, T],$$

where $D_\lambda(\cdot) : \mathbb{R}^n \rightarrow [-1, 1]$ is the dead-zone (set-valued) function defined by

$$\begin{aligned} D_\lambda(w) &= \begin{cases} -1, & \text{if } w < -\lambda, \\ 0, & \text{if } -\lambda < w < \lambda, \\ 1, & \text{if } \lambda < w, \end{cases} \\ D_\lambda(w) &\in [-1, 0], \quad \text{if } w = -\lambda, \\ D_\lambda(w) &\in [0, 1], \quad \text{if } w = \lambda. \end{aligned} \quad (12)$$

See Fig. 1 for the graph of $D_\lambda(\cdot)$.

If $\mathbf{g}_i(\mathbf{x}^*)^\top \mathbf{p}^*$ is equal to $-\lambda_i/T$ or λ_i/T over a non-zero time interval, say $[t_1, t_2] \subset [0, T]$, where $t_1 < t_2$, then the control u_i (and hence \mathbf{u}) over $[t_1, t_2]$ cannot be uniquely determined by the minimum principle. In this case, the interval $[t_1, t_2]$ is called a *singular interval*, and a control problem that has at least one singular interval is called *singular*. If there is no singular interval, the problem is called *normal*:

Definition 7 (Normality): The L^1 -optimal control problem stated in Problem 6 is said to be *normal* if the set

$$\mathcal{T}_i \triangleq \{t \in [0, T] : |T\lambda_i^{-1} \mathbf{g}_i(\mathbf{x}^*(t))^\top \mathbf{p}^*(t)| = 1\}$$

is countable for $i = 1, \dots, m$. If the problem is normal, the elements $t_1, t_2, \dots \in \mathcal{T}_i$ are called the *switching times* for the control $u_i(t)$.

If the problem is normal, the components of the L^1 -optimal control $\mathbf{u}^*(t)$ are piecewise constant and ternary, taking values ± 1 or 0 at almost all⁵ $t \in [0, T]$. This property, named “bang-off-bang,” is the key to relate the L^1 -optimal control with the maximum hands-off control as discussed in the next section.

In general, it is difficult to check if the problem is normal without solving the canonical equations [3, Section 6-22]. For linear plants, however, a sufficient condition for normality is obtained [3, Theorem 6-13].

C. Maximum Hands-Off Control and L^1 Optimality

In this section, we study the relation between maximum hands-off control stated in Problem 4 and L^1 -optimal control stated in Problem 6. The theorem below rationalizes the use of L^1 optimality in computing the maximum hands-off control.

Theorem 8: Assume that the L^1 -optimal control problem (Problem 6) is normal and has at least one solution. Let \mathcal{U}_0^* and \mathcal{U}_1^* be the sets of the optimal solutions of Problem 4 (maximum hands-off control problem) and Problem 6, respectively. Then we have $\mathcal{U}_0^* = \mathcal{U}_1^*$.

Proof: By assumption, \mathcal{U}_1^* is non-empty, and so is $\mathcal{U}(T, \boldsymbol{\xi})$, the set of all admissible controls. Also we have $\mathcal{U}_0^* \subset \mathcal{U}(T, \boldsymbol{\xi})$. We first show that \mathcal{U}_0^* is non-empty, and then prove that $\mathcal{U}_0^* = \mathcal{U}_1^*$.

First, for any $\mathbf{u} \in \mathcal{U}(T, \boldsymbol{\xi})$, we have

$$\begin{aligned} J_1(\mathbf{u}) &= \frac{1}{T} \sum_{i=1}^m \lambda_i \int_0^T |u_i(t)| dt \\ &= \frac{1}{T} \sum_{i=1}^m \lambda_i \int_{\text{supp}(u_i)} |u_i(t)| dt \\ &\leq \frac{1}{T} \sum_{i=1}^m \lambda_i \int_{\text{supp}(u_i)} 1 dt = J_0(\mathbf{u}). \end{aligned} \quad (13)$$

Now take an arbitrary $\mathbf{u}_1^* \in \mathcal{U}_1^*$. Since the problem is normal by assumption, each control $u_{1i}^*(t)$ in $\mathbf{u}_1^*(t)$ takes values -1 , 0, or 1, at almost all $t \in [0, T]$. This implies that

$$\begin{aligned} J_1(\mathbf{u}_1^*) &= \frac{1}{T} \sum_{i=1}^m \lambda_i \int_0^T |u_{1i}^*(t)| dt \\ &= \frac{1}{T} \sum_{i=1}^m \lambda_i \int_{\text{supp}(u_{1i}^*)} 1 dt = J_0(\mathbf{u}_1^*). \end{aligned} \quad (14)$$

From (13) and (14), \mathbf{u}_1^* is a minimizer of J_0 , that is, $\mathbf{u}_1^* \in \mathcal{U}_0^*$. Thus, \mathcal{U}_0^* is non-empty and $\mathcal{U}_1^* \subset \mathcal{U}_0^*$.

Conversely, let $\mathbf{u}_0^* \in \mathcal{U}_0^* \subset \mathcal{U}(T, \boldsymbol{\xi})$. Take independently $\mathbf{u}_1^* \in \mathcal{U}_1^* \subset \mathcal{U}(T, \boldsymbol{\xi})$. From (14) and the optimality of \mathbf{u}_1^* , we have

$$J_0(\mathbf{u}_1^*) = J_1(\mathbf{u}_1^*) \leq J_1(\mathbf{u}_0^*). \quad (15)$$

On the other hand, from (13) and the optimality of \mathbf{u}_0^* , we have

$$J_1(\mathbf{u}_0^*) \leq J_0(\mathbf{u}_0^*) \leq J_0(\mathbf{u}_1^*). \quad (16)$$

It follows from (15) and (16) that $J_1(\mathbf{u}_1^*) = J_1(\mathbf{u}_0^*)$, and hence \mathbf{u}_0^* achieves the minimum value of J_1 . That is, $\mathbf{u}_0^* \in \mathcal{U}_1^*$ and $\mathcal{U}_0^* \subset \mathcal{U}_1^*$. ■

⁵Throughout this paper, “almost all” means “all but a set of Lebesgue measure zero.”

Theorem 8 suggests that L^1 optimization can be used for the maximum hands-off (or the L^0 -optimal) solution. The relation between L^1 and L^0 is analogous to the situation in compressed sensing, where ℓ^1 optimality is often used to obtain the sparsest (i.e. ℓ^0 -optimal) vector; see [15], [16], [23] for details.

Finally, we show that when the system is linear, the number of switching in the maximum hands-off control is bounded.

Proposition 9: Suppose that the plant is given by a linear system

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + \sum_{i=1}^m \mathbf{b}_i u_i(t),$$

where $A \in \mathbb{R}^{n \times n}$ and $\mathbf{b}_1, \dots, \mathbf{b}_m \in \mathbb{R}^n$. Assume that $(A, \mathbf{b}_1), \dots, (A, \mathbf{b}_m)$ are all controllable and A is nonsingular. Assume also that the horizon length $T > 0$ (for given initial state $\mathbf{x}(0) = \boldsymbol{\xi} \in \mathcal{R}$) is chosen such that an L^1 -optimal control exists. Let ω be the largest imaginary part of the eigenvalues of A . Then, the maximum hands-off control is a piecewise constant signal, with values -1 , 0 , and 1 , with no switches from $+1$ to -1 or -1 to $+1$, and with $2nm(1 + T\omega/\pi)$ discontinuities at most.

Proof: Since $(A, \mathbf{b}_1), \dots, (A, \mathbf{b}_m)$ are controllable and A is nonsingular, the L^1 -optimal control problem is normal [3, Theorem 6-13]. Then, by Theorem 8, the maximum hands-off control is identical to the L^1 -optimal control. Combining this with Theorem 3.2 of [21] gives the results. ■

D. Discrete-time hands-off control

Here we consider discrete-time hands-off control. We assume the plant model is given by

$$\mathbf{x}[k+1] = \mathbf{f}(\mathbf{x}[k]) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x}[k])u_i[k], \quad k = 0, 1, \dots, N-1, \quad (17)$$

where $\mathbf{x}[k] \in \mathbb{R}^n$ is the discrete-time state, $u_1[k], \dots, u_m[k]$ are the discrete-time scalar control inputs, \mathbf{f} and \mathbf{g}_i are functions on \mathbb{R}^n . We assume that $\mathbf{f}(\mathbf{x})$, $\mathbf{g}_i(\mathbf{x})$, $\mathbf{f}'(\mathbf{x})$, and $\mathbf{g}'_i(\mathbf{x})$ are continuous. We use the vector notation $\mathbf{u}[k] \triangleq [u_1[k], \dots, u_m[k]]^\top$.

The control $\{\mathbf{u}[0], \mathbf{u}[1], \dots, \mathbf{u}[N-1]\}$ is chosen to drive the state $\mathbf{x}[k]$ from a given initial state $\mathbf{x}[0] = \boldsymbol{\xi}$ to the origin $\mathbf{x}[N] = \mathbf{0}$. The components of the control $\mathbf{u}[k]$ are constrained in magnitude by

$$\max_i |u_i[k]| \leq 1, \quad k = 0, 1, \dots, N-1. \quad (18)$$

We call a control $\{\mathbf{u}[0], \dots, \mathbf{u}[N-1]\}$ admissible (as in the continuous-time case) if it satisfies (18) and the resultant state $\mathbf{x}[k]$ from (17) satisfies $\mathbf{x}[0] = \boldsymbol{\xi}$ and $\mathbf{x}[N] = \mathbf{0}$. We denote by $\mathcal{U}[N, \boldsymbol{\xi}]$ the set of all admissible controls. We assume that N is sufficiently large so that the set $\mathcal{U}[N, \boldsymbol{\xi}]$ is non-empty.

For the admissible control, we consider the discrete-time maximum hands-off control (or ℓ^0 -optimal control) defined by

$$\text{minimize}_{\mathbf{u} \in \mathcal{U}[N, \boldsymbol{\xi}]} J_0(\mathbf{u}), \quad J_0(\mathbf{u}) \triangleq \frac{1}{N} \sum_{i=1}^m \lambda_i \|\mathbf{u}_i\|_{\ell^0}, \quad (19)$$

where $\|\mathbf{v}\|_{\ell^0}$ denotes the number of the nonzero elements of $\mathbf{v} \in \mathbb{R}^N$. The associated ℓ^1 -optimal control problem is given by

$$\begin{aligned} & \text{minimize}_{\mathbf{u} \in \mathcal{U}[N, \boldsymbol{\xi}]} J_1(\mathbf{u}), \\ J_1(\mathbf{u}) & \triangleq \frac{1}{N} \sum_{i=1}^m \lambda_i \|\mathbf{u}_i\|_{\ell^1} = \frac{1}{N} \sum_{i=1}^m \sum_{k=0}^{N-1} \lambda_i |\mathbf{u}_i[k]|. \end{aligned} \quad (20)$$

For the ℓ^1 -optimal control problem, we define the Hamiltonian $H(\mathbf{x}, \mathbf{p}, \mathbf{u})$ by

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}) \triangleq \frac{1}{N} \sum_{i=1}^m \lambda_i |u_i| + \mathbf{p}^\top \left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x})u_i \right),$$

where \mathbf{p} denotes the costate for the ℓ^1 -optimal control problem. Let \mathbf{u}^* be an ℓ^1 -optimal control, and \mathbf{x}^* and \mathbf{p}^* are the associated state and costate, respectively. Then the discrete-time minimum principle [18] gives

$$H(\mathbf{x}^*[k], \mathbf{p}^*[k+1], \mathbf{u}^*[k]) \leq H(\mathbf{x}^*[k], \mathbf{p}^*[k+1], \mathbf{u}[k]),$$

for $k = 0, 1, \dots, N-1$ and all admissible $\mathbf{u} \in \mathcal{U}[N, \boldsymbol{\xi}]$. From this, the ℓ^1 -optimal control \mathbf{u}_i^* (if it exists) satisfies

$$\mathbf{u}_i^*[k] = -D_{\lambda_i/N} \left(\mathbf{g}_i(\mathbf{x}^*[k])^\top \mathbf{p}^*[k+1] \right),$$

where $D_\lambda(\cdot)$ is the dead-zone function defined in (12) (see also Fig. 1). Based on this, we define the discrete-time normality.

Definition 10 (Discrete-time normality): The discrete-time ℓ^1 -optimal control problem is said to be *normal* if

$$|N\lambda_i^{-1} \mathbf{g}_i(\mathbf{x}^*[k])^\top \mathbf{p}^*[k+1]| \neq 1,$$

for $k = 0, 1, \dots, N-1$.

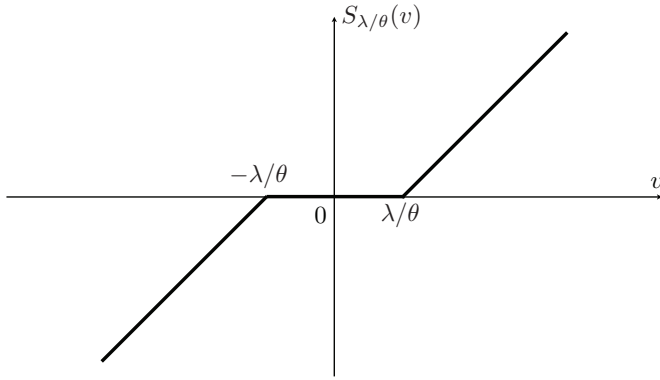
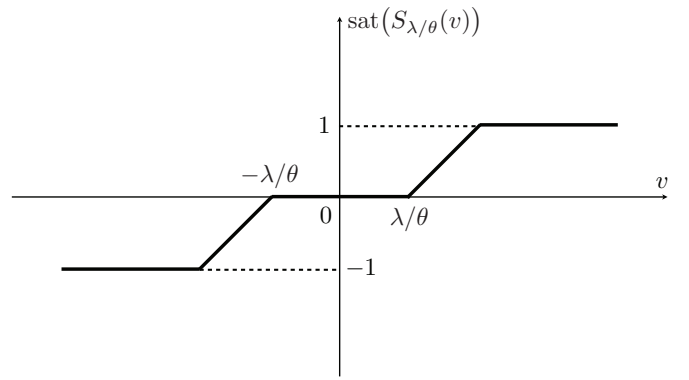
Then we have the following result:

Theorem 11: Assume that the discrete-time ℓ^1 -optimal control problem described in (20) is normal and has at least one solution. Let \mathcal{U}_0^* and \mathcal{U}_1^* be the sets of the solutions of the maximum hands-off control problem in (19) and the ℓ^1 -optimal control problem in (20), respectively. Then we have $\mathcal{U}_0^* = \mathcal{U}_1^*$.

Proof: The theorem can be proved using the same ideas used in the proof of Theorem 8. Details are omitted for sake of brevity. ■

V. L^1/L^2 -OPTIMAL CONTROL

In the previous section, we have shown that the maximum hands-off control problem can be solved via L^1 -optimal control. From the "bang-off-bang" property of the L^1 -optimal control, the control changes its value at switching times *discontinuously*. This is undesirable for some applications in which the actuators cannot move abruptly. In this case, one may want to make the control *continuous*. For this purpose, we add a regularization term to the L^1 cost $J_1(\mathbf{u})$ defined in (10). More precisely, we consider the following mixed L^1/L^2 -optimal control problem.

Fig. 2. Shrinkage function $S_{\lambda/\theta}(v)$ Fig. 3. Saturated shrinkage function $\text{sat}(S_{\lambda/\theta}(v))$

Problem 12 (L^1/L^2 -optimal control): Find an admissible control on $[0, T]$, $\mathbf{u} \in \mathcal{U}(T, \boldsymbol{\xi})$, that minimizes

$$\begin{aligned} J_{12}(\mathbf{u}) &\triangleq \frac{1}{T} \sum_{i=1}^m \left(\lambda_i \|u_i\|_1 + \frac{\theta_i}{2} \|u_i\|_2^2 \right) \\ &= \frac{1}{T} \sum_{i=1}^m \int_0^T \left(\lambda_i |u_i(t)| + \frac{\theta_i}{2} |u_i(t)|^2 \right) dt, \end{aligned} \quad (21)$$

where $\lambda_i > 0$ and $\theta_i > 0$, $i = 1, \dots, m$, are given weights. ■

To discuss the optimal solution(s) of the above problem, we next give necessary conditions for the L^1/L^2 -optimal control using the minimum principle of Pontryagin.

The Hamiltonian function associated to Problem 12 is given by

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}, \mathbf{u}) &= \sum_{i=1}^m \left(\lambda_i |u_i| + \frac{\theta_i}{2} |u_i|^2 \right) \\ &\quad + \mathbf{p}^\top \left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x}) u_i \right) \end{aligned}$$

where \mathbf{p} is the costate vector. Let \mathbf{u}^* denote the optimal control and \mathbf{x}^* and \mathbf{p}^* the resultant optimal state and costate, respectively. Then we have the following result.

Lemma 13: The i -th element $u_i^*(t)$ of the L^1/L^2 -optimal control $\mathbf{u}^*(t)$ satisfies

$$u_i^*(t) = -\text{sat} \left\{ S_{\lambda_i/\theta_i} \left(\theta_i^{-1} \mathbf{g}_i(\mathbf{x}^*(t))^\top \mathbf{p}^*(t) \right) \right\}, \quad (22)$$

where $S_{\lambda/\theta}(\cdot)$ is the shrinkage function defined by

$$S_{\lambda/\theta}(v) \triangleq \begin{cases} v + \lambda/\theta & \text{if } v < -\lambda/\theta, \\ 0, & \text{if } -\lambda/\theta \leq v \leq \lambda/\theta, \\ v - \lambda/\theta, & \text{if } \lambda/\theta < v, \end{cases}$$

and $\text{sat}(\cdot)$ is the saturation function defined by

$$\text{sat}(v) \triangleq \begin{cases} -1, & \text{if } v < -1, \\ v, & \text{if } -1 \leq v \leq 1, \\ 1, & \text{if } 1 < v. \end{cases}$$

See Figs. 2 and 3 for the graphs of $S_{\lambda/\theta}(\cdot)$ and $\text{sat}(S_{\lambda/\theta}(\cdot))$, respectively.

Proof: The result is easily obtained upon noting that

$$-\text{sat} \left\{ S_{\lambda/\theta}(\theta^{-1}a) \right\} = \arg \min_{|u| \leq 1} \lambda |u| + \frac{\theta}{2} |u|^2 + au,$$

for any $\lambda > 0$, $\theta > 0$, and $a \in \mathbb{R}$. ■

From Lemma 13, we have the following proposition.

Proposition 14 (Continuity): The L^1/L^2 -optimal control $\mathbf{u}^*(t)$ is continuous in t over $[0, T]$.

Proof: Without loss of generality, we assume $m = 1$ (a single input plant), and omit subscripts for u , θ , λ , and so on. Let

$$\bar{u}(\mathbf{x}, \mathbf{p}) \triangleq -\text{sat} \left\{ S_{\lambda/\theta}(\theta^{-1} \mathbf{g}(\mathbf{x})^\top \mathbf{p}) \right\}.$$

Since functions $(\text{sat} \circ S_{\lambda/\theta})(\cdot)$ and $\mathbf{g}(\cdot)$ are continuous, $\bar{u}(\mathbf{x}, \mathbf{p})$ is also continuous in \mathbf{x} and \mathbf{p} . It follows from Lemma 13 that the optimal control u^* given in (22) is continuous in \mathbf{x}^* and \mathbf{p}^* . Hence, $u^*(t)$ is continuous, if $\mathbf{x}^*(t)$ and $\mathbf{p}^*(t)$ are continuous in t over $[0, T]$.

The canonical system for the L^1/L^2 -optimal control is given by

$$\begin{aligned} \frac{d\mathbf{x}^*(t)}{dt} &= \mathbf{f}(\mathbf{x}^*(t)) + \mathbf{g}(\mathbf{x}^*(t)) \bar{u}(\mathbf{x}^*(t), \mathbf{p}^*(t)), \\ \frac{d\mathbf{p}^*(t)}{dt} &= -\mathbf{f}'(\mathbf{x}^*(t))^\top \mathbf{p}^*(t) \\ &\quad - \bar{u}(\mathbf{x}^*(t), \mathbf{p}^*(t)) \mathbf{g}'(\mathbf{x}^*(t))^\top \mathbf{p}^*(t). \end{aligned}$$

Since $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$, $\mathbf{f}'(\mathbf{x})$, and $\mathbf{g}'(\mathbf{x})$ are continuous in \mathbf{x} by assumption, and so is $\bar{u}(\mathbf{x}, \mathbf{p})$ in \mathbf{x} and \mathbf{p} , the right hand side of the canonical system is continuous in \mathbf{x}^* and \mathbf{p}^* . From a continuity theorem of dynamical systems, e.g. [3, Theorem 3-14], it follows that the resultant trajectories $\mathbf{x}^*(t)$ and $\mathbf{p}^*(t)$ are continuous in t over $[0, T]$. ■

Proposition 14 motivates us to use the L^1/L^2 optimization in Problem 12 for continuous hands-off control.

In general, the degree of continuity (or smoothness) and the sparsity of the control input cannot be optimized at the same time. The weights λ_i or θ_i can be used for trading smoothness for sparsity. Lemma 13 suggests that increasing the weight λ_i (or decreasing θ_i) makes the i -th input $u_i(t)$ sparser (see also Fig. 3). On the other hand, decreasing λ_i (or increasing θ_i) smoothens $u_i(t)$. In fact, we have the following limiting properties.

Proposition 15 (Limiting cases): Assume the L^1 -optimal control problem is normal. Let $\mathbf{u}_1(\boldsymbol{\lambda})$ and $\mathbf{u}_{12}(\boldsymbol{\lambda}, \boldsymbol{\theta})$ be solutions to respectively Problems 6 and 12 with parameters

$$\boldsymbol{\lambda} \triangleq (\lambda_1, \dots, \lambda_m), \quad \boldsymbol{\theta} \triangleq (\theta_1, \dots, \theta_m).$$

1) For any fixed $\boldsymbol{\lambda} > 0$, we have

$$\lim_{\boldsymbol{\theta} \rightarrow \mathbf{0}} \mathbf{u}_{12}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \mathbf{u}_1(\boldsymbol{\lambda}).$$

2) For any fixed $\boldsymbol{\theta} > 0$, we have

$$\lim_{\boldsymbol{\lambda} \rightarrow \mathbf{0}} \mathbf{u}_{12}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \mathbf{u}_2(\boldsymbol{\theta}),$$

where $\mathbf{u}_2(\boldsymbol{\theta})$ is an L^2 -optimal (or minimum energy) control discussed in [3, Chap. 6], that is, a solution to a control problem where $J_1(\mathbf{u})$ in Problem 6 is replaced with

$$J_2(\mathbf{u}) = \frac{1}{T} \sum_{i=1}^m \frac{\theta_i}{2} \int_0^T |u_i(t)|^2 dt. \quad (23)$$

Proof: The first statement follows directly from the fact that for any fixed $\boldsymbol{\lambda} > 0$, we have

$$\lim_{\boldsymbol{\theta} \rightarrow \mathbf{0}} \text{sat}(S_{\boldsymbol{\lambda}/\boldsymbol{\theta}}(\boldsymbol{\theta}^{-1}\mathbf{w})) = D_{\boldsymbol{\lambda}}(\mathbf{w}), \quad \forall \mathbf{w} \in \mathbb{R} \setminus \{\pm \boldsymbol{\lambda}\},$$

where $D_{\boldsymbol{\lambda}}(\cdot)$ is the dead-zone function defined in (12). The second statement derives from the fact that for any fixed $\boldsymbol{\theta} > 0$, we have

$$\lim_{\boldsymbol{\lambda} \rightarrow \mathbf{0}} \text{sat}(S_{\boldsymbol{\lambda}/\boldsymbol{\theta}}(\mathbf{v})) = \text{sat}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{R}.$$

In summary, the L^1/L^2 -optimal control is an *intermediate control* between the L^1 -optimal control (or the maximum hands-off control) and the L^2 -optimal control.

Example 16: Let us consider the following linear system

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t).$$

We set the final time $T = 10$, and the initial and final states as

$$\mathbf{x}(0) = [1, 1, 1, 1]^\top, \quad \mathbf{x}(10) = \mathbf{0}.$$

Fig. 4 shows the L^1/L^2 optimal control with weights $\lambda_1 = \theta_1 = 1$. The maximum hands-off control is also illustrated. We can see that the L^1/L^2 -optimal control is continuous but sufficiently sparse. Fig. 5 shows the state trajectories of $x_i(t)$, $i = 1, 2, 3, 4$. By the sparse L^1/L^2 control, each state approaches zero within time $T = 10$.

VI. SELF-TRIGGERED HANDS-OFF FEEDBACK CONTROL

In the previous section, we have shown that the maximum hands-off control is given by the solution to an associated L^1 -optimal control problem. The L^1 -optimal control can be computed, for example, via convex optimization after time discretization. However, it is still difficult to give optimal control as a function of the state variable $\mathbf{x}(t)$. This is a drawback if there exist uncertainties in the plant model and disturbances

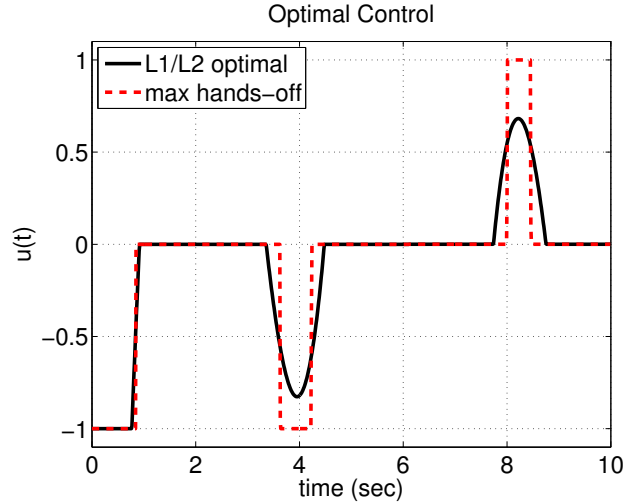


Fig. 4. Maximum hands-off control (dashed) and L^1/L^2 -optimal control (solid)

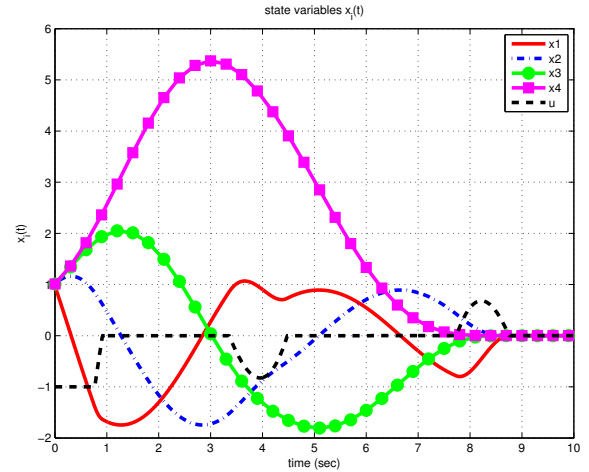


Fig. 5. State trajectory by L^1/L^2 -optimal control

added to the signals. Therefore, we extend maximum hands-off control to feedback control. In this section, we assume the controlled plant model is given by a single-input, linear time-invariant system

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + \mathbf{b}u(t) + \mathbf{d}(t), \quad t \in [0, \infty), \quad (24)$$

where $A \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$ are given constants, and $\mathbf{d}(t) \in \mathbb{R}^n$ denotes an unknown plant disturbance. For a nonlinear plant, one can use (24) as a linearized model and $\mathbf{d}(t)$ as the linearization error (see Section VII). We assume that

- 1) (A, \mathbf{b}) is reachable,
- 2) A is nonsingular.

This is a sufficient condition so that the L^1 -optimal control problem with the single-input linear system (24) in the disturbance-free case where $\mathbf{d} \equiv \mathbf{0}$ is normal for any horizon length $T > 0$ and any initial condition $\mathbf{x}(0) \in \mathcal{R}$ [3, Theorem 6-13].

A. Sparsity Rate for Infinite Horizon Signals

Before considering feedback control, we define the sparsity rate for infinite horizon signals (cf. Definition 1).

Definition 17 (Sparsity rate): For infinite horizon signal $u = \{u(t) : t \in [0, \infty)\}$, we define the sparsity rate by

$$R_\infty(u) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \|u|_{[0, T]}\|_0, \quad (25)$$

where $u|_{[0, T]}$ is the restriction of u to the interval $[0, T]$. Note that

- 1) If $\|u\|_0 < \infty$, then $R_\infty(u) = 0$.
- 2) If $|u(t)| > 0$ for almost all $t \in [0, \infty)$, then $R_\infty(u) = 1$.
- 3) For any measurable function u on $[0, \infty)$, we have $0 \leq R_\infty(u) \leq 1$.

We say again that an infinite horizon signal u is *sparse* if the sparsity rate $R_\infty(u) \ll 1$.

Lemma 18: Let u be a measurable function on $[0, \infty)$. If there exist time instants t_0, t_1, t_2, \dots such that

$$\begin{aligned} t_0 = 0, \quad t_{k+1} = t_k + T_k, \quad T_k > 0, \\ R_{T_k}(u|_{[t_k, t_{k+1}]}) \leq r, \quad \forall k \in \{0, 1, 2, \dots\}, \end{aligned}$$

then $R_\infty(u) \leq r$.

Proof: The following calculation proves the statement.

$$\begin{aligned} R_\infty(u) &= \lim_{T \rightarrow \infty} \frac{1}{T} \|u|_{[0, T]}\|_0 \\ &= \lim_{N \rightarrow \infty} \frac{1}{t_N} \sum_{k=0}^{N-1} \|u|_{[t_k, t_{k+1}]}\|_0 \\ &= \lim_{N \rightarrow \infty} \frac{1}{t_N} \sum_{k=0}^{N-1} (t_{k+1} - t_k) R_{T_k}(u|_{[t_k, t_{k+1}]}) \\ &\leq \lim_{N \rightarrow \infty} \frac{1}{t_N} (t_N - t_0) r \\ &= r \end{aligned}$$

B. Control Algorithm

Fix a bound on the sparsity rate $R_\infty(u) \leq r$ with $r \in (0, 1)$. We here propose a feedback control algorithm that achieves the sparsity rate r of the resultant control input. Our method involves applying maximum hands-off control over finite horizons, and to use self-triggered feedback to compensate for disturbances. In self-triggered control, the next update time is determined by the current plant state.

First, let us assume that an initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ is given. For this, we compute the minimum-time $T^*(\mathbf{x}_0)$, the solution of the minimum-time control. Then, we define the first sampling period (or the first horizon length) by

$$T_0 \triangleq \max \{T_{\min}, r^{-1}T^*(\mathbf{x}_0)\},$$

where T_{\min} is a given positive time length that prevents the sampling period from zero (thereby avoiding Zeno executions [43]). For this horizon length, we compute the maximum hands-off control on the interval $[0, T_0]$. Let this optimal control be denoted $u_0(t)$, $t \in [0, T_0]$, that is

$$u_0(t) = \arg \min_{u \in \mathcal{U}(T_0, \mathbf{x}_0)} \|u\|_0, \quad t \in [0, T_0],$$

Algorithm 1 Self-triggered Hands-off Control

Given initial state \mathbf{x}_0 and minimum inter-sampling time T_{\min} .

Let $\mathbf{x}(0) = \mathbf{x}_0$ and $t_0 = 0$.

for $k = 0, 1, 2, \dots$ **do**

 Measure $\mathbf{x}_k := \mathbf{x}(t_k)$.

 Compute $T^*(\mathbf{x}_k)$.

 Put $T_k := \max \{T_{\min}, r^{-1}T^*(\mathbf{x}_k)\}$.

 Put $t_{k+1} := t_k + T_k$.

 Compute max hands-off control

$$u_k = \arg \min_{u \in \mathcal{U}(T_k, \mathbf{x}_k)} \|u\|_0. \quad (27)$$

 Apply $u(t) = u_k(t - t_k)$, $t \in [t_k, t_{k+1}]$ to the plant.

end for

where $\mathcal{U}(T_0, \mathbf{x}_0)$ is the set of admissible control on time interval $[0, T_0]$ with initial state \mathbf{x}_0 ; see Section III. Apply this control, $u_0(t)$, to the plant (24) from $t = 0$ to $t = T_0$. If $d \equiv 0$ (i.e. no disturbances), then $\mathbf{x}(T_0) = 0$ by the terminal constraint, and applying $u(t) = 0$ for $t \geq T_0$ gives $\mathbf{x}(t) = 0$ for all $t \geq T_0$.

However, if $d \neq 0$, then $\mathbf{x}(T_0)$ will in general not be exactly zero. To steer the state to the origin, we should again apply a control to the plant. Let $\mathbf{x}_1 \triangleq \mathbf{x}(T_0)$, and $t_1 \triangleq T_0$. We propose to compute the minimum time $T^*(\mathbf{x}_1)$ and let

$$T_1 \triangleq \max \{T_{\min}, r^{-1}T^*(\mathbf{x}_1)\}.$$

For this horizon length T_1 , we compute the maximum hands-off control, $u_1(t)$, $t \in [t_1, t_1 + T_1]$, as well, which is applied to the plant on the time interval $[t_1, t_1 + T_1]$.

Continuing this process gives a self-triggered feedback control algorithm, described in Algorithm 1, which results in an infinite horizon control

$$u(t) = u_k(t - t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \dots, \quad (26)$$

where u_k is defined in (27). For this control, we have the following proposition.

Proposition 19 (Sparsity rate): For the infinite horizon control u in (26), the sparsity rate $R_\infty(u)$ is less than r .

Proof: Fix $k \in \{0, 1, 2, \dots\}$. Let $\mathbf{x}_k \triangleq \mathbf{x}(t_k)$. The k -th horizon length T_k is given by

$$T_k = \max \{T_{\min}, r^{-1}T^*(\mathbf{x}_k)\}. \quad (28)$$

Let us first consider the case when $T_{\min} \leq r^{-1}T^*(\mathbf{x}_k)$, or $T_k = r^{-1}T^*(\mathbf{x}_k)$. Let $u_k^*(t)$ denote the minimum-time control for initial state \mathbf{x}_k , and define

$$\tilde{u}_k(t) := \begin{cases} u_k^*(t), & t \in [0, T^*(\mathbf{x}_k)], \\ 0, & t \in (T^*(\mathbf{x}_k), r^{-1}T^*(\mathbf{x}_k)]. \end{cases} \quad (29)$$

Note that $T^*(\mathbf{x}_k) < r^{-1}T^*(\mathbf{x}_k)$ since $r \in (0, 1)$. Clearly this is an admissible control, that is, $\tilde{u}_k \in \mathcal{U}(T_k, \mathbf{x}_k)$, and

$$\|\tilde{u}_k\|_0 = \|u_k^*\|_0 = T^*(\mathbf{x}_k),$$

for which see also Fig. 6. On the other hand, let u_k denote

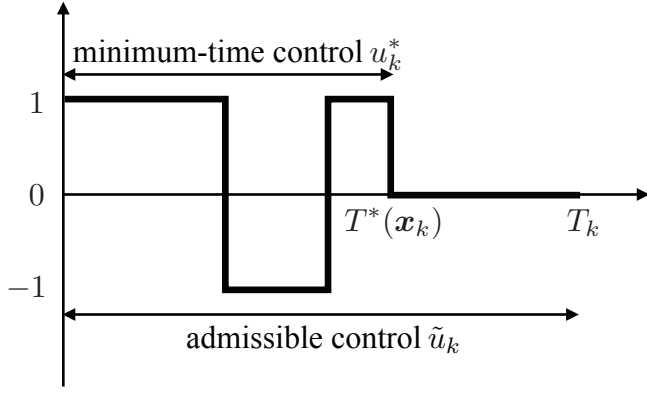


Fig. 6. Minimum-time control $u_k^*(t)$ and admissible control $\tilde{u}_k(t)$ defined in (29).

the maximum hands-off control on time interval $[0, T_k]$ with initial state \mathbf{x}_k . Since u_k has the minimum L^0 norm, we have

$$\|u_k\|_0 \leq \|\tilde{u}_k\|_0 = T^*(\mathbf{x}_k).$$

It follows that the sparsity rate of $u_k(t - t_k)$, $t \in [t_k, t_k + T_k]$ is

$$R_{T_k}(u_k) = \frac{1}{T_k} \|u_k\|_0 \leq \frac{T^*(\mathbf{x}_k)}{r^{-1}T^*(\mathbf{x}_k)} = r.$$

Next, for the case when $T_{\min} \geq r^{-1}T^*(\mathbf{x}_k)$, we have $T_{\min} > T^*(\mathbf{x})$. It follows that $R_{T_k}(u_k) \leq r$ by a similar argument. In either case, we have $R_{T_k}(u_k) \leq r$ for $k = 0, 1, 2, \dots$. Finally, Lemma 18 gives the result. ■

Remark 20 (Minimum time computation): Algorithm 1 includes computation of the minimum time $T^*(\mathbf{x}_k)$. For single-input, linear time-invariant system, an efficient numerical algorithm has been proposed in [10], which one can use for the computation. Also, this can be used to check whether the initial state \mathbf{x}_0 lies in the reachable set \mathcal{R} .

C. Practical Stability

By the feedback control algorithm (Algorithm 1), the state $\mathbf{x}(t)$ is sampled at sampling instants t_k , $k = 1, 2, \dots$, and between sampling instants the system acts as an open loop system. Since there exists disturbance $\mathbf{d}(t)$, it is impossible to asymptotically stabilize the feedback system to the origin. We thus focus on *practical stability* of the feedback control system under bounded disturbances. The following are fundamental lemmas to prove the stability.

Lemma 21: For $A \in \mathbb{R}^{n \times n}$, we have

$$\|e^{At}\| \leq e^{\mu(A)t}, \quad \forall t \in [0, \infty),$$

where $\mu(A)$ is the maximum eigenvalue of $(A + A^\top)/2$, that is,

$$\mu(A) = \lambda_{\max} \left(\frac{A + A^\top}{2} \right). \quad (30)$$

Proof: This can be easily proved by a general theorem of the matrix measure [11, Theorem II.8.27]. ■

Lemma 22: There exists a scalar-valued, continuous, and non-decreasing function $\alpha : [0, \infty) \rightarrow [0, \infty)$ such that

- 1) $\alpha(0) = 0$,

- 2) $T^*(\mathbf{x}) \leq \alpha(\|\mathbf{x}\|)$, $\forall \mathbf{x} \in \mathcal{R}$, where \mathcal{R} is the reachable set defined in Definition 3

Proof: For $v \geq 0$, define

$$\alpha(v) \triangleq \max_{\|\xi\| \leq v} T^*(\xi).$$

By this definition, it is easy to see that if $v_1 \geq v_2$ then $\alpha(v_1) \geq \alpha(v_2)$. Since $T^*(\xi)$ is continuous on \mathcal{R} (see [20]), $\alpha(v)$ is continuous. The first statement is a result from $T^*(\mathbf{0}) = 0$. Then, setting $v = \|\mathbf{x}\|$ for $\mathbf{x} \in \mathcal{R}$ gives the second statement. ■

Now, we have the following stability theorem.

Theorem 23: Assume that the plant noise is bounded by $\delta > 0$, that is, $\|\mathbf{d}(t)\| \leq \delta$ for all $t \geq 0$. Assume also that the initial state $\mathbf{x}(0) = \mathbf{x}_0$ is in the reachable set \mathcal{R} , and let

$$T_0 \triangleq \max\{T_{\min}, r^{-1}T^*(\mathbf{x}_0)\}. \quad (31)$$

Define

$$\begin{aligned} \Omega &\triangleq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq \gamma\}, \\ \gamma &\triangleq \frac{\delta}{\mu(A)} \left(e^{\mu(A)T_0} - 1 \right), \end{aligned} \quad (32)$$

and assume $\Omega \subset \mathcal{R}$. Choose a function α which satisfies the conditions in Lemma 22. If

$$\alpha(\gamma) \leq rT_0, \quad (33)$$

then the feedback control with Algorithm 1 achieves practical stability in the sense that

- 1) $\mathbf{x}(t)$ is bounded for $t \in [0, t_1]$.
- 2) $\mathbf{x}_k \triangleq \mathbf{x}(t_k) \in \Omega$, $\forall k \in \{1, 2, \dots\}$.
- 3) For $t \in [t_k, t_{k+1}]$, $k \in \{1, 2, \dots\}$, we have $\|\mathbf{x}(t)\| \leq h$, where if $\mu(A) < 0$

$$h = \gamma + \frac{\|\mathbf{b}\| + \delta}{|\mu(A)|} \triangleq h_1,$$

and if $\mu(A) > 0$

$$h = h_1 e^{\mu(A) \max\{T_{\min}, r^{-1}\alpha(\gamma)\}} - \frac{\|\mathbf{b}\| + \delta}{\mu(A)}.$$

Proof: Since the system is linear time-invariant and $u(t)$ and $\mathbf{d}(t)$ are bounded, the state $\mathbf{x}(t)$ is also bounded on $[0, t_1]$. For $t = t_1$, we have

$$\begin{aligned} \|\mathbf{x}_1\| = \|\mathbf{x}(t_1)\| &\leq \int_0^{T_0} \|e^{A(T_0-\tau)}\| \delta d\tau \\ &\leq \int_0^{T_0} e^{\mu(A)(T_0-\tau)} \delta d\tau \\ &= \frac{\delta}{\mu(A)} \left(e^{\mu(A)T_0} - 1 \right), \end{aligned}$$

and hence $\mathbf{x}_1 = \mathbf{x}(t_1) \in \Omega$. Note that since $\mathbf{x}_0 \in \mathcal{R}$, we have $T_0 < \infty$. Note also that since A is nonsingular, $\mu(A) \neq 0$. Fix $k \in \{1, 2, \dots\}$, and assume $\mathbf{x}_k = \mathbf{x}(t_k) \in \Omega$. Then we have

$$\|\mathbf{x}_{k+1}\| \leq \frac{\delta}{\mu(A)} \left(e^{\mu(A)T_k} - 1 \right),$$

where T_k is as in (28). Note that $T_k < \infty$ since $\mathbf{x}_k \in \Omega \subset \mathcal{R}$. If $T_k = T_{\min}$ then

$$\begin{aligned} \|\mathbf{x}_{k+1}\| &\leq \frac{\delta}{\mu(A)} \left(e^{\mu(A)T_{\min}} - 1 \right) \\ &\leq \frac{\delta}{\mu(A)} \left(e^{\mu(A)T_0} - 1 \right) = \gamma \end{aligned}$$

since $T_0 \geq T_{\min}$. On the other hand, if $T_k = r^{-1}T^*(\mathbf{x}_k)$ then

$$\begin{aligned} \|\mathbf{x}_{k+1}\| &\leq \frac{\delta}{\mu(A)} \left(e^{\mu(A)r^{-1}T^*(\mathbf{x}_k)} - 1 \right) \\ &= \gamma + \frac{\delta}{\mu(A)} \left(e^{\mu(A)r^{-1}T^*(\mathbf{x}_k)} - e^{\mu(A)T_0} \right). \end{aligned} \quad (34)$$

Lemma 22, assumption $\mathbf{x}_k \in \Omega$, and equation (33) give

$$T^*(\mathbf{x}_k) \leq \alpha(\|\mathbf{x}_k\|) \leq \alpha(\gamma) \leq rT_0, \quad (35)$$

and hence

$$e^{\mu(A)r^{-1}T^*(\mathbf{x}_k)} - e^{\mu(A)T_0} \leq 0.$$

From (34), we have $\|\mathbf{x}_{k+1}\| \leq \gamma$. In each case, we have $\mathbf{x}_{k+1} = \mathbf{x}(t_{k+1}) \in \Omega$.

Then, let us consider the intersample behavior of $\mathbf{x}(t)$, $t \in [t_k, t_{k+1}]$ for $k = 1, 2, \dots$. As proved above, we have $\mathbf{x}_k = \mathbf{x}(t_k) \in \Omega$. This gives

$$\begin{aligned} \|\mathbf{x}(t)\| &\leq \|e^{A(t-t_k)}\| \|\mathbf{x}_k\| + \int_{t_k}^t \|e^{A(t-\tau)}\| \|\mathbf{b}\| |u_k(t)| d\tau \\ &\quad + \int_{t_k}^t \|e^{A(t-\tau)}\| \|\mathbf{d}(\tau)\| d\tau \\ &\leq e^{\mu(A)(t-t_k)} \|\mathbf{x}_k\| + \int_{t_k}^t e^{\mu(A)(t-\tau)} d\tau (\|\mathbf{b}\| + \delta) \\ &= e^{\mu(A)(t-t_k)} \|\mathbf{x}_k\| + \frac{\|\mathbf{b}\| + \delta}{\mu(A)} \left(e^{\mu(A)(t-t_k)} - 1 \right). \end{aligned}$$

If $\mu(A) < 0$ then $\mathbf{x}(t)$ is bounded as

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}_k\| + \frac{\|\mathbf{b}\| + \delta}{|\mu(A)|} \leq \gamma + \frac{\|\mathbf{b}\| + \delta}{|\mu(A)|}.$$

If $\mu(A) > 0$ then $\mathbf{x}(t)$ is again bounded as

$$\begin{aligned} \|\mathbf{x}(t)\| &\leq e^{\mu(A)T_k} \|\mathbf{x}_k\| + \frac{\|\mathbf{b}\| + \delta}{\mu(A)} \left(e^{\mu(A)T_k} - 1 \right) \\ &\leq e^{\mu(A) \max\{T_{\min}, r^{-1}\alpha(\gamma)\}} \gamma \\ &\quad + \frac{\|\mathbf{b}\| + \delta}{\mu(A)} \left(e^{\mu(A) \max\{T_{\min}, r^{-1}\alpha(\gamma)\}} - 1 \right). \end{aligned}$$

From (31) and (32), we conclude that the larger the sparsity rate r , the smaller the upper bound γ . This shows there is a tradeoff between the sparsity rate of control and the performance. The analysis is deterministic and the bound is for the worst-case disturbance, but this is reasonably tight in some cases when a worst-case disturbance is applied to the system, as shown in the example below. ■

VII. EXAMPLE

First, let us consider a simple example with a 1-dimensional stable plant model

$$\frac{dx(t)}{dt} = ax(t) + au(t) + d(t), \quad (36)$$

where $a < 0$. We assume bounded disturbance, that is, there exists $\delta > 0$ such that $|d(t)| \leq \delta$ for all $t \geq 0$. The plant is normal and hence the maximum hands-off control is given by L^1 -optimal control thanks to Theorem 8. In fact, the optimal control u_k in (27) is computed via the minimum principle for L^1 -optimal control [3, Section 6.14] as

$$u_k(t) = \begin{cases} 0, & t \in [0, \tau), \\ -\text{sgn}(x(t_k)), & t \in [\tau, T_k], \end{cases}$$

where

$$\tau \triangleq \frac{1}{|a|} \log \left(e^{|a|T_k} - |x(t_k)| \right).$$

Also, the minimum time function $T^*(x)$ is computed as (see [3, Example 6-4])

$$T^*(x) = \frac{1}{|a|} \log(1 + |x|), \quad x \in \mathbb{R}.$$

It follows that the reachable set $\mathcal{R} = \mathbb{R}$, and the condition $\Omega \subset \mathcal{R}$ in Theorem 23 always holds. Since $A = a \in \mathbb{R}$, we have $\mu(A) = a$ by (30). Then, for any $x \in \mathbb{R}$, we have

$$T^*(x) = \frac{1}{|a|} \log(1 + |x|) \leq \frac{|x|}{|a|},$$

and hence we can choose $\alpha(v) = v/|a|$ for Lemma 22 and Theorem 23. The stability condition (33) becomes

$$\alpha \left(\frac{\delta}{|a|} (1 - e^{aT_0}) \right) \leq rT_0$$

or $rT_0 a^2 \geq \delta(1 - e^{aT_0})$.

For example, with $a = -1$, $\delta = 1$, $x_0 = 1$, and if we choose $T_{\min} < T^*(x_0) = \log(1 + |x_0|) = \log 2$, then $rT_0 = \log 2$ and the condition becomes

$$r \geq -\frac{\log 2}{\log(1 - \log 2)} \approx 0.587.$$

We set $r = 0.6$ and simulate the feedback control with disturbance $d(t)$ as uniform noise with mean 0 and bound $\delta = 1$. Fig. 7 shows the maximum hands-off control obtained by Algorithm 1. We can observe that the control is sufficiently sparse. In fact, the sparsity rate for this control is $R_\infty(u) = 0.148$, which is smaller than the upper bound $r = 0.6$.

Since the plant is asymptotically stable, one can choose the zero control, that is, $u \equiv 0$, to achieve stability, which is the sparsest. Fig. 8 shows the state $x(t)$ for the maximum hands-off control and the zero control. Due to the time optimality of the hands-off control, the state approaches to 0 faster than that of the zero control.

Then let us consider the influence of disturbances. The bound γ in (32) is computed as $\gamma = 1 - \exp(-r^{-1} \log 2)$ with $r = 0.6$, and the set Ω becomes

$$\Omega = \{x \in \mathbb{R} : |x| \leq 1 - \exp(-r^{-1} \log 2)\}.$$

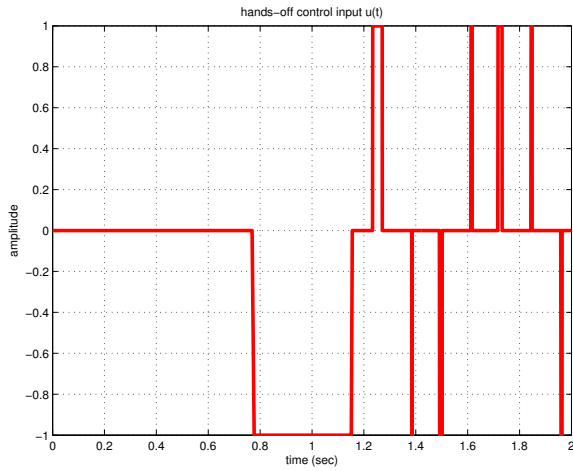
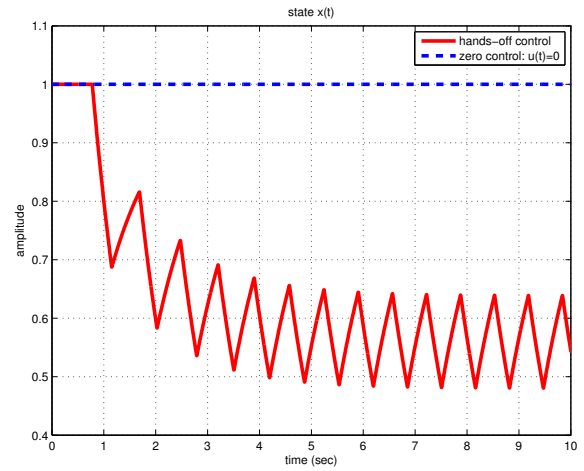
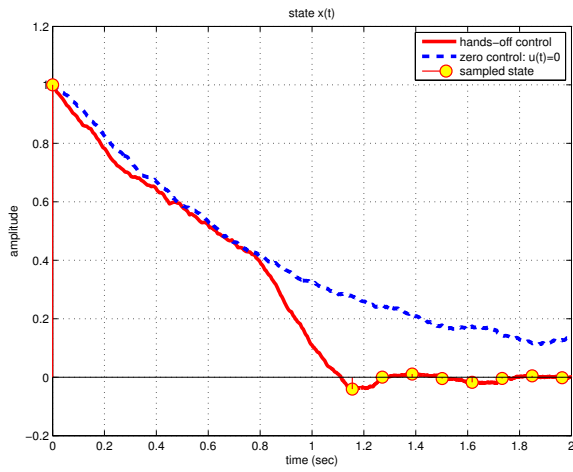
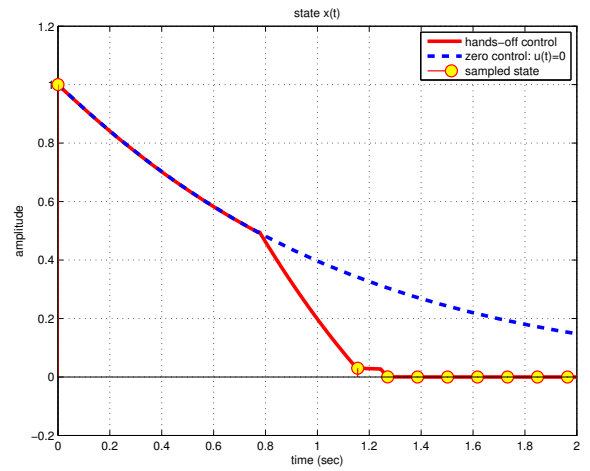
Fig. 7. Hands-off feedback control with sparsity rate $R_\infty(u) = 0.148$.

Fig. 9. State trajectory with worst-case disturbance: hands-off control (solid) and zero control (dots).

Fig. 8. State trajectory: hands-off control (solid) and zero control (dots). Sampled states $x(t_k)$ are also shown (circles)Fig. 10. State trajectory of nonlinear plant (37) with $a = -1$ (stable): hands-off control (solid), zero control (dots), and sampled states $x(t_k)$ (circles).

This bound is obtained in a deterministic manner, and hence the bound is for the worst-case disturbance. In fact, let us apply a worst-case disturbance $d(t) = 1$ for all $t \geq 0$ to the feedback system. Fig. 9 shows the state trajectories. The trajectory by the zero control remains 1 and do not approach 0, while that by the maximum hands-off control still approaches 0, and we can see that the bound is reasonably tight.

Next, let us consider a nonlinear plant model

$$\frac{dx(t)}{dt} = \sin(ax(t)) + au(t). \quad (37)$$

We linearize this nonlinear plant to obtain the linear plant (36), with the linearization error $d(t) \triangleq \sin(ax(t)) - ax(t)$. Assume $a = -1$ (i.e. stable). We adopt the control law given as above to the nonlinear plant (37). Fig. 10 shows the result. This figure shows that the hands-off control works well for the nonlinear plant (37). The sparsity rate of the hands-off control is $R_\infty(u) = 0.0717$, which is sufficiently small.

On the other hand, let us consider the nonlinear plant (37) with $a = 1$ (i.e. unstable). For the linearized plant (37), the

hands-off control law is given by

$$u_k(t) = \begin{cases} -\text{sgn}(x(t_k)), & t \in [0, \tau), \\ 0, & t \in [\tau, T_k], \end{cases}$$

where $\tau \triangleq -a^{-1} \log(1 - |x(t_k)|)$. The minimum time function $T^*(x)$ is given by $T^*(x) = -a^{-1} \log(1 - |x|)$ for $x \in \mathcal{R}$, where $\mathcal{R} = (-1, 1)$. We set the initial state $x_0 = 0.25$ and the sparsity rate $r = 0.6$, and simulate the feedback control with the nonlinear plant (37). Fig. 11 shows the obtained state trajectory of (37). Obviously, the zero control cannot stabilize the unstable plant and hence the state diverges, while the hands-off control keeps the state close to the origin. The sparsity rate is $R_\infty(u) = 0.1135$, which is sufficiently small.

VIII. CONCLUSION

In this paper, we have proposed maximum hands-off control. It has the minimum support per unit time, or is the sparsest, among all admissible controls. Under normality assumptions,

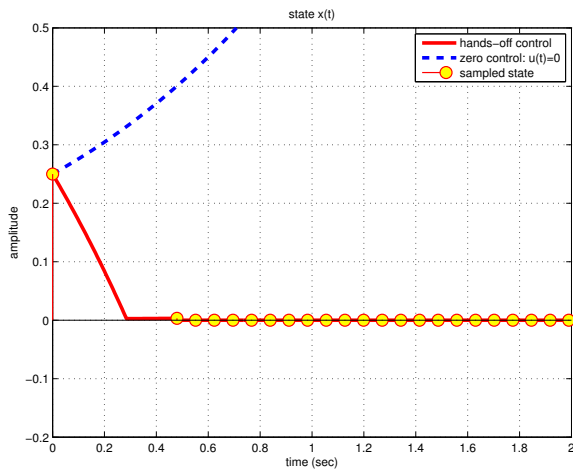


Fig. 11. State trajectory of the nonlinear plant (37) with $\alpha = 1$ (unstable): hands-off control (solid), zero control (dots), and sampled states $x(t_k)$ (circles).

the maximum hands-off control can be computed via L^1 -optimal control. For linear systems, we have also proposed a feedback control algorithm, which guarantees a given sparsity rate and practical stability. An example has illustrated the effectiveness of the proposed control. Future work includes the development of an effective computation algorithm for maximum hands-off control, for situations when the control problem does not satisfy normality conditions, and also when the plant is nonlinear.

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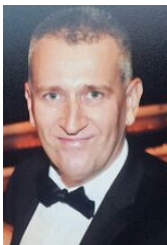


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