Value Function in Maximum Hands-off Control for Linear Systems ⋆

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Abstract

In this brief paper, we study the value function in maximum hands-off control. Maximum hands-off control, also known as sparse control, is the L^0-optimal control among the feasible controls. Although the L^0 measure is discontinuous and non-convex, we prove that the value function, or the minimum L^0 norm of the control, is a continuous and strictly convex function of the initial state in the reachable set, under an assumption on the controlled plant model. We then extend the finite-horizon maximum hands-off control to model predictive control (MPC), and prove the recursive feasibility and the stability by using the continuity and convexity properties of the value function.

Key words: Optimal control, continuity, bang-bang control, discontinuous control, linear systems, minimum-time control, predictive control.

1 Introduction

Optimal control is widely used in recent industrial products not just for achieving the best performance but for reducing the control effort. For example, the classical LQR (Linear Quadratic Regulator) control gives a way to consider the tradeoff between performance and control-effort reduction by using weighting functions on the states and the control inputs with the L^2 norm (i.e. the energy); see Anderson and Moore (2007) for example.

Recently, a novel control method, called *maximum hands-off control*, has been proposed in Nagahara et al. (2013, 2016), which maximizes the time duration in which the control is exactly zero among the feasible controls. An example of hands-off control is a stop-start system in automobiles, in which an automobile automatically shuts down the engine (i.e. zero control) to avoid it idling for long periods of time, and also to reduce CO or CO2 emissions as well as fuel consumption. Therefore, the hands-off control is a kind of *green control* as discussed in Nagahara et al. (2014b). Also, the hands-off control is effective in hybrid/electric vehicles, railway vehicles, networked/embedded systems, to name a few; see Nagahara et al. (2016).

Maximum hands-off control is related to *sparsity*, which is widely studied in compressed sensing, for which we point the reader to Eldar and Kutyniok (2012). Sparsity is also applied to control problems such as networked control in Nagahara et al. (2014a); Kong et al. (2015), security of control systems in Fawzi et al. (2014), state estimation in Sanandaji et al. (2014), to name a few.

A mathematical difficulty in the maximum hands-off control is that the cost function, which is defined by the L^0 measure (the support length of a function), is highly nonlinear; it is discontinuous and non-convex. To solve this problem, a recent work of Nagahara et al. (2013, 2016) has proposed to reduce the problem to an L^1 optimal control problem, and shown the equivalence between the maximum hands-off (or L^0 optimal) control and the L^1 optimal control under the assumption of normality.

Motivated by this work, we investigate the value function in the maximum hands-off control. The value function is defined as the optimal value of the cost function of the optimal control problem. Although the L^0 measure in the maximum hands-off control is discontinuous and non-convex, we prove that the value function is a continuous and strictly convex function of the initial state in the reachable set, under an assumption on the controlled plant model. We then extend the finite-horizon maximum hands-off control to model predictive control

⋆ This paper was not presented at any IFAC meeting. Corresponding author M. Nagahara.

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(MPC), and prove the recursive feasibility (see Rossiter (2004)) and the stability by using the continuity and convexity properties of the value function.

The present paper expands on our recent conference contributions of Ikeda and Nagahara (2015b,a) by rearranging the contents, incorporating convexity analysis of the value function, and including extension to model predictive control.

The remainder of this paper is organized as follows: In Section 2, we give mathematical preliminaries for our subsequent discussion. In Section 3, we review the problem of maximum hands-off control. Section 4 investigates the continuity of the value function in maximum hands-off control, and Section 5 discusses its convexity. Section 6 discusses model predictive control and the stability. Section 7 presents an example of model predictive control to illustrate the effectiveness of the proposed method. In Section 8, we offer concluding remarks.

2 Mathematical Preliminaries

This section reviews basic definitions, facts, and notation that will be used throughout the paper.

Let n be a positive integer. For a vector $x \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$, the *ε-neighborhood* of x is defined by

$$\mathcal{B}(x, \varepsilon) \triangleq \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Let $\mathcal{X}$ be a subset of $\mathbb{R}^n$. A point $x \in \mathcal{X}$ is called an *interior point* of $\mathcal{X}$ if there exists $\varepsilon > 0$ such that $\mathcal{B}(x, \varepsilon) \subset \mathcal{X}$. The *interior* of $\mathcal{X}$ is the set of all interior points of $\mathcal{X}$, and we denote the interior of $\mathcal{X}$ by $\mathrm{int}\mathcal{X}$. A set $\mathcal{X}$ is said to be *open* if $\mathcal{X} = \mathrm{int}\mathcal{X}$. For example, $\mathrm{int}\mathcal{X}$ is open for every subset $\mathcal{X} \subset \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is called an *adherent point* of $\mathcal{X}$ if $\mathcal{B}(x, \varepsilon) \cap \mathcal{X} \neq \emptyset$ for every $\varepsilon > 0$, and the *closure* of $\mathcal{X}$ is the set of all adherent points of $\mathcal{X}$. A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *closed* if $\mathcal{X} = \overline{\mathcal{X}}$, where $\overline{\mathcal{X}}$ is the closure of $\mathcal{X}$. The *boundary* of $\mathcal{X}$ is the set of all points in the closure of $\mathcal{X}$, not belonging to the interior of $\mathcal{X}$, and we denote the boundary of $\mathcal{X}$ by $\partial\mathcal{X}$, i.e., $\partial\mathcal{X} = \overline{\mathcal{X}} - \mathrm{int}\mathcal{X}$, where $\mathcal{X}_1 - \mathcal{X}_2$ is the set of all points which belong to the set $\mathcal{X}_1$ but not to the set $\mathcal{X}_2$. In particular, if $\mathcal{X}$ is closed, then $\partial\mathcal{X} = \mathcal{X} - \mathrm{int}\mathcal{X}$, since $\mathcal{X} = \overline{\mathcal{X}}$. A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *convex* if, for any $x, y \in \mathcal{X}$ and any $\lambda \in [0, 1]$, $(1 - \lambda)x + \lambda y$ belongs to $\mathcal{X}$.

A real-valued function f defined on $\mathbb{R}^n$ is said to be *upper semi-continuous* on $\mathbb{R}^n$ if for every $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : f(x) < \alpha\}$ is open, and f is said to be *lower semi-continuous* on $\mathbb{R}^n$ if for every $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : f(x) > \alpha\}$ is open. It is known that a function f is continuous on $\mathbb{R}^n$ if and only if it is upper and lower semi-continuous on $\mathbb{R}^n$; see e.g. (Rudin, 1987, pp. 37). A real-valued function f defined on a convex set $\mathcal{C} \subset \mathbb{R}^n$ is said to be *convex* if

$$f\big((1 - \lambda)x + \lambda y\big) \leq (1 - \lambda)f(x) + \lambda f(y),$$

for all x, $y \in \mathcal{C}$ and all $\lambda \in (0, 1)$, and f is said to be *strictly convex* if the above inequality holds strictly whenever x and y are distinct points and $\lambda \in (0, 1)$.

Let $T > 0$. For a continuous-time signal $u(t)$ over a time interval $[0, T]$, we define its *L^1 and L^∞ norms* respectively by

$$\|u\|_1 \triangleq \int_0^T |u(t)|dt, \quad \|u\|_\infty \triangleq \sup_{t \in [0,T]} |u(t)|.$$

We define the support set of u, denoted by $\mathrm{supp}(u)$, by the closure of the set $\{t \in [0, T] : u(t) \neq 0\}$. The *$L^0$ norm* of a measurable function u as the length of its support, that is,

$$\|u\|_0 \triangleq m\big(\mathrm{supp}(u)\big),$$

where m is the Lebesgue measure on $\mathbb{R}$.

3 Maximum Hands-off Control Problem

In this paper, we consider a linear time-invariant system represented by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times 1}$. We here consider a single-input case for simplicity (see Ikeda and Nagahara (2015b) for a multi-input case). Throughout this paper, we assume the following:

Assumption 1 *The pair (A, B) is controllable and the matrix A is nonsingular.*

Let $T > 0$ be the final time of control. For the system (1), we call a control $u = \{u(t) : t \in [0, T]\} \in L^1$ *feasible* if it steers $x(t)$ from a given initial state $x(0) = \xi \in \mathbb{R}^n$ to the origin at time T (i.e., $x(T) = 0$), and satisfies the magnitude constraint $\|u\|_\infty \leq 1$. We denote by $\mathcal{U}(\xi)$ the set of all feasible controls for an initial state $\xi \in \mathbb{R}^n$, that is,

$$\mathcal{U}(\xi) \triangleq \left\{u \in L^1 : \int_0^T e^{-As}Bu(s)ds = -\xi, \|u\|_\infty \leq 1\right\}. \tag{2}$$

The *maximum hands-off control* is the minimum L^0-norm (or the sparsest) control among the feasible control inputs. This control problem is formulated as follows.

Problem 2 (Maximum hands-off control) *For a given initial state $\xi \in \mathbb{R}^n$, find a feasible control $u \in \mathcal{U}(\xi)$ that minimizes $J(u) = \|u\|_0$.*

Fig. 1. The L^0 kernel $\phi_0(u)$ and its convex approximation $|u|$ for the L^1 norm.

The value function for this optimal control problem is defined as

$$V(\xi) \triangleq \min_{u \in \mathcal{U}(\xi)} J(u) = \min_{u \in \mathcal{U}(\xi)} \|u\|_0. \tag{3}$$

Note that the cost function $J(u)$ can be rewritten as

$$J(u) = \int_0^T \phi_0(u)\ dt,$$

where ϕ_0 is the L^0 kernel function defined by

$$\phi_0(u) \triangleq \begin{cases} 1, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Fig. 1 shows the graph of $\phi_0(u)$. As shown in this figure, the kernel function $\phi_0(u)$ is discontinuous at $u = 0$ and non-convex. However, in the following sections, we will show that the value function $V(\xi)$ in (3) is continuous and strictly convex.

4 Continuity of Value Function

In this section, we investigate the continuity of the value function $V(\xi)$ in (3).

First, we define the *reachable set* for the control problem (Problem 2) by

$$\mathcal{R} \triangleq \left\{ \int_0^T e^{-As} Bu(s) ds : \|u\|_\infty \leq 1 \right\} \subset \mathbb{R}^n.$$

The following is a fundamental lemma of the paper:

Lemma 3 *Suppose Assumption 1 is satisfied. Let us consider L^1 optimal control with*

$$J_1(u) := \int_0^T |u(t)| dt, \quad V_1(\xi) := \min_{u \in \mathcal{U}(\xi)} J_1(u). \tag{4}$$

Then, for every $\xi \in \mathcal{R}$, we have $V(\xi) = V_1(\xi)$.

PROOF. By Assumption 1, the L^1-optimal control problem associate with (4) is normal (see (Athans and Falb, 1966, Theorem 6-13)). Also, for $\xi \in \mathcal{R}$, an L^1-optimal control $u^* \in \mathcal{U}(\xi)$ minimizing J_1 exists (see Lemma 12 in Appendix A), and $u^*(t) \in \{-1, 0, 1\}$ for almost all $t \in [0, T]$ (this is called the *"bang-off-bang" property*; see (Athans and Falb, 1966, Section 6-14) for example). Then by (Nagahara et al., 2013, Theorem 5), u^* is also the optimal control of Problem 2, and we have

$$V(\xi) = \min_{u \in \mathcal{U}(\xi)} \|u\|_0 = \|u^*\|_0 = \|u^*\|_1 = V_1(\xi),$$

where we used the "bang-off-bang" property of u^* for the third equality. □

Note that the absolute value $|u|$ in (4) is a convex approximation of $\phi_0(u)$ as shown in Fig. 1. Associated with $V_1(\xi)$, we define the following subset of $\mathcal{R}$ with $\alpha \geq 0$:

$$\mathcal{R}_\alpha \triangleq \left\{ \int_0^T e^{-As} Bu(s) ds : \|u\|_\infty \leq 1, \|u\|_1 \leq \alpha \right\}. \tag{5}$$

For the set $\mathcal{R}_\alpha$, we have another fundamental lemma.

Lemma 4 *Suppose Assumption 1 is satisfied. Then, for every $\alpha \in [0, T]$,*

$$\mathcal{R}_\alpha = \{\xi \in \mathcal{R} : V(\xi) \leq \alpha\}, \tag{6}$$

$$\partial\mathcal{R}_\alpha = \{\xi \in \mathcal{R} : V(\xi) = \alpha\}, \tag{7}$$

$$\text{int}\mathcal{R}_\alpha = \{\xi \in \mathcal{R} : V(\xi) < \alpha\}. \tag{8}$$

PROOF. See Appendix A. □

From these lemmas, we show the continuity of the value function $V(\xi)$.

Theorem 5 *If Assumption 1 is satisfied, then $V(\xi)$ is continuous on $\mathcal{R}$.*

PROOF. Define

$$\overline{V}(\xi) \triangleq \begin{cases} V(\xi), & \text{if } \xi \in \mathcal{R}, \\ T, & \text{if } \xi \in \mathbb{R}^n - \mathcal{R}. \end{cases}$$

It is enough to show that $\overline{V}(\xi)$ is continuous on $\mathbb{R}^n$.

First, we show that the lower level set

$$\mathcal{L}_\alpha \triangleq \{\xi \in \mathbb{R}^n : \overline{V}(\xi) < \alpha\}$$

is open for every $\alpha \in \mathbb{R}$. If $\alpha \leq 0$, then the set $\mathcal{L}_\alpha$ is empty since for any $\xi \in \mathbb{R}^n$, $\overline{V}(\xi) \geq 0$. If $\alpha > T$, then the set $\mathcal{L}_\alpha$ is $\mathbb{R}^n$, since for any $\xi \in \mathcal{R}$, $V(\xi) \leq T$. If $0 < \alpha \leq T$, then the set $\mathcal{L}_\alpha$ is a subset of $\mathcal{R}$, and coincides with $\mathrm{int}\mathcal{R}_\alpha$ by Lemma 4. Therefore, the set $\mathcal{L}_\alpha$ is open for every $\alpha \in \mathbb{R}$. It follows that $V(\xi)$ is upper semi-continuous on $\mathbb{R}^n$.

Next, we show that the upper level set

$$\mathcal{L}^\alpha \triangleq \{\xi \in \mathbb{R}^n : \overline{V}(\xi) > \alpha\}$$

is open for every $\alpha \in \mathbb{R}$. If $\alpha < 0$ or $\alpha \geq T$, then the set $\mathcal{L}^\alpha$ is $\mathbb{R}^n$ or empty, respectively. If $0 \leq \alpha < T$, from Lemma 4, we have

$$\mathcal{L}^\alpha = \mathbb{R}^n - \{\xi \in \mathcal{R} : V(\xi) \leq \alpha\} = \mathbb{R}^n - \mathcal{R}_\alpha.$$

Since $\mathcal{R}_\alpha$ is closed (see Lemma 10 in Appendix A), the set $\mathcal{L}^\alpha$ is open for every $\alpha \in \mathbb{R}$. It follows that $\overline{V}(\xi)$ is lower semi-continuous on $\mathbb{R}^n$.

Since $\overline{V}(\xi)$ is upper and lower semi-continuous on $\mathbb{R}^n$, it is continuous on $\mathbb{R}^n$, and the conclusion follows. □

Theorem 5 leads to an important result of L^1 optimal control as follows.

Corollary 6 *If Assumption 1 is satisfied, then $V_1(\xi)$ is continuous on $\mathcal{R}$.*

PROOF. This is a direct consequence of Lemma 3 and Theorem 5. □

5 Convexity of Value Function

Here we show the convexity of the value function $V(\xi)$. Although the kernel function $\phi_0(u)$ in the cost function is not convex as shown in Fig. 1, the value function $V(\xi)$ is a convex function on $\mathcal{R}$.

Theorem 7 *If Assumption 1 is satisfied, then $V(\xi)$ is strictly convex on $\mathcal{R}$.*

PROOF. From Lemma 3, it is enough to prove that the L^1 value function $V_1(\xi)$ is strictly convex on $\mathcal{R}$

First, we prove that $V_1(\xi)$ is convex on $\mathcal{R}$. Take any ξ, $\eta \in \mathcal{R}$, and $\lambda \in (0,1)$. Then there exist L^1-optimal controls u_ξ and u_η for initial states ξ and η, respectively (see Lemma 12 in Appendix A). Obviously, the following control

$$u \triangleq (1-\lambda)u_\xi + \lambda u_\eta \tag{9}$$

steers the state from the initial state $(1-\lambda)\xi + \lambda\eta$ to the origin at time T, and it satisfies $\|u\|_\infty \leq 1$. That is, we have $u \in \mathcal{U}\big((1-\lambda)\xi + \lambda\eta\big)$. Therefore

$$\begin{aligned} V_1\big((1-\lambda)\xi + \lambda\eta\big) &\leq \|u\|_1 \\ &\leq (1-\lambda)\|u_\xi\|_1 + \lambda\|u_\eta\|_1 \\ &= (1-\lambda)V_1(\xi) + \lambda V_1(\eta), \end{aligned} \tag{10}$$

and hence $V_1(\xi)$ is convex on $\mathcal{R}$.

Next, we will show the strict convexity of $V(\xi)$. To prove this, we will show that a contradiction is implied by assuming that there exist ξ, $\eta \in \mathcal{R}$ with $\xi \neq \eta$ and $\lambda \in (0,1)$ such that

$$V_1\big((1-\lambda)\xi + \lambda\eta\big) = (1-\lambda)V_1(\xi) + \lambda V_1(\eta). \tag{11}$$

Let u_ξ and u_η be L^1-optimal controls for initial states ξ and η, respectively. Let u be as in (9). From (10) and (11), it follows that

$$V_1\big((1-\lambda)\xi + \lambda\eta\big) = \|u\|_1 = (1-\lambda)\|u_\xi\|_1 + \lambda\|u_\eta\|_1,$$

so the control $u = (1-\lambda)u_\xi + \lambda u_\eta$ is an L^1-optimal control for the initial state $(1-\lambda)\xi + \lambda\eta$.

Now, by Assumption 1, $u_\xi(t)$ and $u_\eta(t)$ take the values 1, 0, and -1 at almost all $t \in [0,T]$. So, the pair $(u_\xi(t), u_\eta(t))$ takes the following values on $[0,T]$ except for sets of measure zero: $(1,1)$, $(1,0)$, $(1,-1)$, $(0,1)$, $(0,0)$, $(0,-1)$, $(-1,1)$, $(-1,0)$, $(-1,-1)$. For the above pairs of $(u_\xi(t), u_\eta(t))$, the control $u = (1-\lambda)u_\xi + \lambda u_\eta$ respectively takes the following values: 1, $1-\lambda$, $1-2\lambda$, λ, 0, $-\lambda$, $-1+2\lambda$, $-1+\lambda$, -1. On the other hand, the control u is also L^1 optimal and takes the values 1, 0, and -1 at almost all $t \in [0,T]$. Since $\lambda \in (0,1)$, we have

$$m(\mathcal{I}_{1,0} \cup \mathcal{I}_{0,1} \cup \mathcal{I}_{0,-1} \cup \mathcal{I}_{-1,0}) = 0, \tag{12}$$

where

$$\mathcal{I}_{i,j} \triangleq \{t \in [0,T] : (u_\xi(t), u_\eta(t)) = (i,j)\},$$

for $i, j \in \{-1, 0, 1\}$. If $\lambda \neq 1/2$, then we also have

$$m(\mathcal{I}_{1,-1} \cup \mathcal{I}_{-1,1}) = 0,$$

and it follows that

$$m(\mathcal{I}_{1,1} \cup \mathcal{I}_{0,0} \cup \mathcal{I}_{-1,-1}) = T,$$

that is, $u_\xi(t) = u_\eta(t)$ for almost all $t \in [0,T]$. This implies $\xi = \eta$, but this contradicts the assumption, so we have $\lambda = 1/2$. Then the pair $(u_\xi(t), u_\eta(t))$ on $[0,T]$ except for sets of measure zero takes values $(1,1)$, $(1,-1)$, $(0,0)$, $(-1,1)$, and $(-1,-1)$. Since $\xi \neq \eta$, we have

$$T_1 \triangleq m(\mathcal{I}_{1,-1} \cup \mathcal{I}_{-1,1}) > 0. \tag{13}$$

Let $T_2 \triangleq m(\mathcal{I}_{1,1})$ and $T_3 \triangleq m(\mathcal{I}_{-1,-1})$. From (12) and the fact that $u_\xi + u_\eta = 0$ on $\mathcal{I}_{1,-1} \cup \mathcal{I}_{-1,1} \cup \mathcal{I}_{0,0}$, we have

$$V_1\left(\frac{1}{2}\xi + \frac{1}{2}\eta\right) = \frac{1}{2}\|u_\xi + u_\eta\|_1 = T_2 + T_3, \qquad (14)$$

On the other hand,

$$\begin{aligned}\frac{1}{2}V_1(\xi) + \frac{1}{2}V_1(\eta) &= \frac{1}{2}\|u_\xi\|_1 + \frac{1}{2}\|u_\eta\|_1 \\ &= T_1 + T_2 + T_3.\end{aligned} \qquad (15)$$

Equations (11), (14) and (15) imply that $T_1 = 0$, which contradicts (13). $\square$

6 Model Predictive Control

In this section, we extend the sparse optimal control to model predictive control (MPC). We assume the controlled plant is given by (1). Here we assume state feedback, that is, for every $t \in [0, \infty)$, $x(t)$ can be obtained.

The control algorithm adopted here is given as follows. First, let us take a sequence $\{t_k\}_{k=0}^\infty$ of sampling times, where we assume $0 = t_0 < t_1 < t_2 < \cdots$ and there exists $\tau > 0$ such that

$$\tau < t_{k+1} - t_k \leq T, \quad k = 0, 1, 2, \ldots. \qquad (16)$$

We also assume that an initial state $x(0) = \xi \in \mathcal{R}$ is given. At each sampling time t_k, the optimal control $u_k(t)$ for $t \in [t_k, t_k + T]$ is computed by solving the sparse optimal control problem (Problem 2) on the interval $[t_k, t_k + T]$, where the current state $x(t_k)$ is used as the initial state, and only the control on the interval $[t_k, t_{k+1}]$ is applied to the plant. If each optimization has the optimal solution, then this process gives a control

$$u(t) = u_k(t), \quad t \in [t_k, t_{k+1}], \qquad (17)$$

where $k = 0, 1, 2, \ldots$. Since $t_{k+1} - t_k \geq \tau > 0$, the control $u(t)$ is defined for all $t \in [0, \infty)$.

We first show the recursive feasibility, which is discussed in Rossiter (2004), of the MPC, that is, for $t = t_1, t_2, \ldots$, the optimal control problem is feasible. Equivalently, we show $x(t_k) \in \mathcal{R}$ for $k = 1, 2, \ldots$.

Theorem 8 *Assume* (16) *is satisfied and* $x(0) \in \mathcal{R}$. *If Assumption 1 is satisfied, then we have* $x(t_k) \in \mathcal{R}$ *for* $k = 1, 2, \ldots$.

PROOF. If Assumption 1 is satisfied, then from Lemma 12 in Appendix 12 and Theorem 8 of Nagahara et al. (2016), the sparse optimal control exists for any initial states in the reachable set $\mathcal{R}$. The state $x(t)$ for $t \in [0, t_1]$ obviously exists in the reachable set $\mathcal{R}$ while the control u_0 is used, since any point out of the set $\mathcal{R}$ needs a time duration more than T to be steered to the origin by any control v with $\|v\|_\infty \leq 1$. Therefore the next optimization at the time t_1 has the optimal solution u_1, and then the state $x(t)$ for $t \in [t_1, t_2]$ exists in the reachable set $\mathcal{R}$ while the control u_1 is used. It follows that the state exists in the reachable set $\mathcal{R}$ at all times under this situation, in particular $x(t_k) \in \mathcal{R}$ for $k = 1, 2, \ldots$. $\square$

By this theorem, the optimal control u_k always exists and hence the control u is well defined.

Next, we investigate the stability under the model predictive control given in (17). More precisely, the question here is whether the origin is stable in the sense of Lyapunov regardless of how to take sampling times when we use the control u defined by (17). From the recursive feasibility and the continuity and convexity of the value function V, we can prove the origin is stable in the sense of Lyapunov, that is, when a sequence $\{t_k\}_{k=0}^\infty$ is taken, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial state ξ with $\|\xi\| < \delta$ an inequality $\|x(t)\| < \varepsilon$ holds for all $t \geq 0$, where x is the state with $x(0) = \xi$ and is obtained by using the control u given in (17).

Theorem 9 *Assume* (16) *is satisfied and* $x(0) \in \mathcal{R}$. *If Assumption 1 is satisfied, then the origin is stable in the sense of Lyapunov regardless of how the sequence* $\{t_k\}_{k=0}^\infty$ *of sampling times is taken when the control* u *defined by* (17) *is used.*

PROOF. Fix a sequence $\{t_k\}_{k=0}^\infty$ and $\varepsilon > 0$. Since $\mathcal{R}$ contains the origin as an interior point if the pair (A, B) is controllable (see (Hermes and Lasalle, 1969, Theorem 17.3)), we can take $r \in (0, \varepsilon)$ such that

$$\mathcal{B}_r \triangleq \{\xi \in \mathbb{R}^n : \|\xi\| \leq r\} \subset \mathcal{R}.$$

Since V is continuous on $\partial\mathcal{B}_r$, we can define

$$\alpha \triangleq \min_{\|\xi\|=r} V(\xi).$$

Clearly, $\alpha > 0$ since $V(\xi) > 0$ for $\xi \neq 0$. Take $\beta \in (0, \alpha)$, then the set $\mathcal{R}_\beta \cap \partial\mathcal{B}_r$ is empty, and the set $\mathcal{R}_\beta$ is convex from the convexity of V and Lemma 4, and it contains the origin. Therefore we have $\mathcal{R}_\beta \subset \mathrm{int}\mathcal{B}_r$.

From the continuity of V at the origin, there exists $\delta > 0$ such that $\|\xi\| < \delta$ implies

$$0 \leq V(\xi) \leq \beta. \qquad (18)$$

When we use the control u defined by (17) for ξ with $\|\xi\| < \delta$, it is clear that we have

$$V(x_\xi(t)) \leq V(\xi) \tag{19}$$

for all $t \geq 0$, where $x_\xi(t)$ is the state with $x_\xi(0) = \xi$ and is obtained by using the control u. Therefore for ξ with $\|\xi\| < \delta$ we have $V(x_\xi(t)) \leq \beta$ for all $t \geq 0$ from (18) and (19). Since $\mathcal{R}_\beta \subset \mathcal{B}_r \subset \mathcal{B}_\varepsilon$, for any initial state ξ with $\|\xi\| < \delta$ we have $x_\xi(t) \in \mathcal{B}_\varepsilon$ for all $t \geq 0$, which means that the origin is stable in the sense of Lyapunov. $\square$

7 Example

In this section, we consider a simple example with a 1-dimensional (i.e. $n = 1$) linear control system (1) with $A = a > 0$ (unstable) and $B = b \neq 0$. The pair (A, B) obviously satisfies Assumption 1, and hence the value function $V(\xi)$ is continuous and convex on the reachable set $\mathcal{R}$ (see below for details).

The reachable set $\mathcal{R}$ and the maximum hands-off control u_ξ for an initial state $\xi \in \mathcal{R}$ are computed via the bang-bang principle (see (Hermes and Lasalle, 1969, Theorem 12.1)) and the minimum principle for L^1-optimal control (see (Athans and Falb, 1966, Section 6.14)) as

$$\mathcal{R} = [-x_1, x_1], \quad x_1 = \frac{|b|}{a}(1 - e^{-aT}),$$

and

$$u_\xi(t) = \begin{cases} -\mathrm{sgn}(b)\mathrm{sgn}(\xi), & t \in [0, \tau_\xi), \\ 0, & t \in [\tau_\xi, T], \end{cases}$$

where $\mathrm{sgn}(x) = x/|x|$ for $x \neq 0$ and $\mathrm{sgn}(0) = 0$, and

$$\tau_\xi \triangleq \frac{1}{a} \log \frac{|b|}{|b| - a|\xi|}.$$

On the other hand, the conventional energy-minimizing control (i.e. L^2-optimal control) that minimizes the L^2 norm of the control u subject to $u \in \mathcal{U}(\xi)$ is given by

$$v_\xi(t) = \begin{cases} -\mathrm{sgn}(b)\mathrm{sgn}(\xi), & t \in [0, \theta_\xi), \\ c(\xi)e^{-at}, & t \in [\theta_\xi, T], \end{cases}$$

for some $c(\xi) \in \mathbb{R}$ and $\theta_\xi > 0$ (see (Athans and Falb, 1966, Section 6.20)).

Using these controls, we simulate MPC with $a = 1$, $b = 2$, $T = 5$, and $\xi = 1$. In this case, $x_1 = 2(1 - e^{-5})$, and the value function is obtained by

$$V(\xi) = \|u_\xi\|_0 = |\tau_\xi| = \left|\log \frac{2}{2 - |\xi|}\right|,$$

Fig. 2. Value function $V(\xi)$

for $\xi \in \mathcal{R} = [-x_1, x_1]$. Fig. 2 shows the value function, which is clearly continuous and convex. The sampling instants for MPC are taken as $t_1 = 2$, $t_2 = 5.5$, $t_3 = 6.5$, and $t_4 = 10$. Under these parameters, we consider a noisy control system

$$\dot{x}(t) = ax(t) + bu(t) + w(t),$$

where $w(t)$ is additive noise generated from the uniform distribution on $[-1, 1]$. Fig. 3 shows the simulation results (control $u(t)$ and state $x(t)$) with the two controls. We can see that the L^0 MPC gives a sparser control than the L^2 MPC, while the state trajectory by L^0 MPC is more attenuated than that by L^2 MPC.

8 Conclusion

In this brief paper, we have proved the continuity and the strict convexity of the value function of the maximum hands-off control problem under an assumption of the controlled system. These properties of the value function plays an important role to prove the stability when we extend the control to the model predictive control.

Acknowledgements

This research was supported in part by JSPS KAKENHI Grant Numbers 26120521, 15K14006, and 15H02668.

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Fig. 3. Control $u(t)$ (top) and state $x(t)$ (bottom) with L^0 MPC (solid) and L^2 MPC (dashed)

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A Proof of Lemma 4

To prove Lemma 4, we need some lemmas.

Lemma 10 *The set R_α in* (5) *satisfies the following:*

(1) For every $\alpha \in \mathbb{R}$, $\mathcal{R}_\alpha$ is compact.
(2) For every $\alpha \in \mathbb{R}$, $\mathcal{R}_\alpha \subset \mathcal{R}$, with equality for $\alpha \geq T$.
(3) $\mathcal{R}_0 = \{0\}$.
(4) $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$ for $0 \leq \alpha \leq \beta$.

PROOF. See (Hajek, 1979, Lemma 2.1). □

Lemma 11 *For every $\alpha \in [0, T]$, we have*

$$\mathcal{R}_\alpha = \{\xi \in \mathcal{R} : \exists\, u \in \mathcal{U}(\xi) \text{ s.t. } \|u\|_1 \leq \alpha\}.$$

PROOF. First, fix $\alpha \in [0, T]$ and take any $\xi \in \mathcal{R}_\alpha$. Then, by the definition of $\mathcal{R}_\alpha$, there exists $u \in \mathcal{U}(\xi)$ such that $\|u\|_1 \leq \alpha$ and

$$\xi = \int_0^T e^{-As} Bu(s) ds.$$

From (2), it follows that the control $v := -u$ is a feasible control, that is, $v \in \mathcal{U}(\xi)$, and also satisfies $\|v\|_1 = \|u\|_1 \leq \alpha$. By definition, $\mathcal{R}_\alpha \subset \mathcal{R}$ and hence $\xi \in \mathcal{R}$. Therefore, we have

$$\xi \in \{\xi \in \mathcal{R} : \exists\, u \in \mathcal{U}(\xi) \text{ s.t. } \|u\|_1 \leq \alpha\}.$$

Conversely, fix $\alpha \in [0, T]$ and take any

$$\xi \in \{\xi \in \mathcal{R} : \exists\, u \in \mathcal{U}(\xi) \text{ s.t. } \|u\|_1 \leq \alpha\}.$$

That is, $\xi \in \mathcal{R}$ is an initial state for the system (1), and there exists a feasible control $u \in \mathcal{U}(\xi)$ such that $\|u\|_1 \leq \alpha$. Then from (2), we have

$$\xi = \int_0^T e^{-As} B\big(-u(s)\big) ds.$$

The control $v = -u$ satisfies

$$\|v\|_1 = \|u\|_1 \leq \alpha, \quad \|v\|_\infty = \|u\|_\infty \leq 1,$$

and hence we have $\xi \in \mathcal{R}_\alpha$. $\square$

Lemma 12 *For each initial value $\xi \in \mathcal{R}$, there exists a feasible control $u \in \mathcal{U}(\xi)$ with minimum L^1-cost $\|u\|_1$. Furthermore, then, $\xi \in \partial\mathcal{R}_\alpha$ with $\alpha = \|u\|_1$.*

PROOF. See (Hajek, 1979, Lemma 3.1). $\square$

Now, let us prove (6). First, fix $\alpha \in [0, T]$ and take any $\xi \in \mathcal{R}_\alpha$. Then, from Lemma 10, we have $\xi \in \mathcal{R}$, and from Lemma 12, there exists an L^1-optimal control $u^* \in \mathcal{U}(\xi)$. Also, we have $V_1(\xi) = \|u^*\|_1 \leq \alpha$ by Lemma 11. Then, from Lemma 3, we have $V(\xi) \leq \alpha$. That is, we have

$$\xi \in \overline{\mathcal{L}}_\alpha \triangleq \{\xi \in \mathcal{R} : V(\xi) \leq \alpha\}.$$

Conversely, fix $\alpha \in [0, T]$ and take any $\xi \in \overline{\mathcal{L}}_\alpha$. From Lemma 3, we have $V_1(\xi) \leq \alpha$. Let $\beta \triangleq V_1(\xi)$. From Lemma 12, we have $\xi \in \partial\mathcal{R}_\beta$, and it follows from Lemma 10 that $\xi \in \partial\mathcal{R}_\beta \subset \mathcal{R}_\beta \subset \mathcal{R}_\alpha$.

Next, we prove the equation (7); then the equation (8) follows immediately from (6) and (7), since $\mathcal{R}_\alpha$ is closed for every $\alpha \geq 0$ from Lemma 10. If $\alpha = 0$, then $\partial\mathcal{R}_0 = \{0\}$, since $\mathcal{R}_0 = \{0\}$. It follows from (6) that

$$\{\xi \in \mathcal{R} : V(\xi) = 0\} = \mathcal{R}_0 = \{0\} = \partial\mathcal{R}_0.$$

Fix $\alpha \in (0, T]$. We can take $\xi \in \partial\mathcal{R}_\alpha$, since $\partial\mathcal{R}_\alpha$ is not empty (Note that $\mathbb{R}^n$ and the empty set are the only subsets whose boundaries are empty, since $\mathbb{R}^n$ is connected; see (Singh, 2013, Chapter 3)). Since $\xi \in \mathcal{R}_\alpha$, we have $V(\xi) \leq \alpha$. If $V(\xi) < \alpha$, then from (Hajek, 1979, Lemma 4.2) we have

$$\xi \in \partial\mathcal{R}_{V(\xi)} \subset \mathcal{R}_{V(\xi)} \subset \mathrm{int}\mathcal{R}_\alpha,$$

and hence a contradiction occurs. Therefore we have $V(\xi) = \alpha$, and hence

$$\partial\mathcal{R}_\alpha \subset \{\xi \in \mathcal{R} : V(\xi) = \alpha\},$$

and $\{\xi \in \mathcal{R} : V(\xi) = \alpha\}$ is not empty for every $\alpha \in (0, T]$. Then it follows from Lemma 12 that

$$\{\xi \in \mathcal{R} : V(\xi) = \alpha\} \subset \partial\mathcal{R}_\alpha$$

for every $\alpha \in (0, T]$, and the conclusion follows.