

## A BRIEF OVERVIEW OF SIGNAL RECONSTRUCTION VIA SAMPLED-DATA $H^\infty$ OPTIMIZATION

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**ABSTRACT.** This paper gives a brief account of a new method for signal reconstruction by employing the mathematical machinery from sampled-data control theory. We formulate the signal reconstruction problem in terms of an analog performance optimization problem using a stable discrete-time filter. The proposed  $H^\infty$  performance criterion naturally takes intersample behavior into account, reflecting the energy distributions of the signal. We present a method for computing optimal solutions which are guaranteed to be stable and causal. Comparisons to alternative methods are also presented.

**Keywords:** Sampled-Data Control Theory, Digital Signal Processing, Shannon Paradigm, Sampling Theorem,  $H^\infty$  Control Theory.

**AMS Subject Classification:** 93C57, 60G35, 93B36, 94A20.

### 1. INTRODUCTION

Digitization of analog information such as voice, images, and video is a ubiquitous feature of modern communications and information technology. Therefore, reconstruction of the original analog signal from its digital version is a problem of great interest and importance. These reconstruction procedures are used everywhere: music players, mobile telephones, cameras, television, etc. We note that there is a critical distinction between compression/transmission/recovery of digital data and our focus here with the reconstruction of the original analog data from which such digital data are generated.

Shannon's pioneering and much celebrated paper [27] was a landmark development in this signal reconstruction problem. Under the assumption that the original analog signal is band-limited (below the so-called Nyquist frequency), he showed that the original analog signal can be exactly reconstructed by using the sampling theorem [44]. Shannon's framework established a fundamental paradigm for digital signal processing. We will hereafter refer to this scheme the Shannon paradigm. However, the band-limitedness assumption, necessary for perfect signal reconstruction, is not easily satisfied. In many applications, the sampling rate is not high enough to allow for this assumption to hold even approximately. To address this problem, an anti-aliasing filter is often introduced to sharply cut high frequency components. This in turn leads to yet another type of distortion due to the Gibbs phenomenon (see Section 6 below). Moreover, the sinc function, which is the impulse response of the ideal reconstruction filter, is not causal and does not decay very fast. This slow decay rate makes it very difficult to implement. Various approximations thus become necessary. This procedure further complicates the digital filter design procedure, making it less transparent.

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Motivated by these issues, many researchers have been developing techniques that aim to solve the signal reconstruction problem under the assumption of non-ideal signal acquisition devices. An excellent survey of this stream of literature can be found in [30]. In this research framework, the digital signal is assumed to be the sampled version of an analog signal processed through a non-ideal analog filter. The reconstruction process then attempts to recover the original signal. The idea is to place the problem into the framework of the (orthogonal or oblique) projection theorem (in usually  $L^2$ ), and then project the signal space to the subspace generated by the shifted reconstruction functions. It is often required that the process give a consistent result, i.e., if we subject the reconstructed signal to the whole sampling process again, it should yield the same sampled values from which it was reconstructed [28].

The principal objective of the present paper is to present a relatively brief summary of an entirely new approach to the signal reconstruction problem. This approach takes its inspiration from the modern sampled-data control theory, developed in the control community since the 1990's. The fundamental accomplishment of modern sampled-data control theory is that it gives us a discrete-time controller (or filter) that optimizes the closed-loop performance with intersample behavior taken into account. In particular, it optimizes an analog (continuous-time) performance metric. These metrics are given in terms of  $H^\infty$  or  $H^2$  norms. This framework gives us an ideal platform to reconstruct the original analog signals from their sampled-data versions when original signal is not band-limited.

Chen and Francis [5] made a first attempt to apply sampled-data control theory to signal processing (however in a discrete-time domain); see also [14]. Starting in 1995, the present authors and our colleagues have pursued the signal reconstruction problem in the sampled-data context to obtain an optimal analog performance via digital filtering: See [19, 41, 36, 42] for general design frameworks, [15, 25] for sample-rate conversion, [40] for multirate filterbank design, [1, 2] for audio signal compression, [16] for image restoration, [26] for fractional delay filters, [17] for wavelet expansion, and [35, 39] for convergence analysis. The method has also been patented [37, 38, 12, 13] and implemented into sound processing LSI chips as a core technology by Sanyo Semiconductors, and successfully used in mobile phones, digital voice recorders and MP3 players; their cumulative production has exceeded 30 million units as of 2011. Our objective here is to present a succinct exposition of some key aspects of this body of work.

The same philosophy of emphasizing the importance of analog performance was proposed and pursued recently by Unser and co-workers [31, 32]. The crucial difference is however that they rely on  $L^2/H^2$  type optimization and oblique projections, which are very different from our method here. In particular, it can raise some stability questions. The recent work of Meinsma and Mirkin [21, 22] takes an approach that is close to ours. They give solutions for non-causal problems and allow freedom in the choice of sample or hold devices. A detailed comparison of our work and these related works is provided in [42]. Some other approaches (not very closely related to our work) to extending the traditional sampling theory include: reconstruction by quasi-interpolation [6], and minimization of the worst-case regret [9].

The present paper is organized as follows: After preparing some basic notions in function spaces in Section 2, we first review the fundamentals in signal reconstruction using the sampling theorem, and discuss its various drawbacks in Section 3. We will then give a fundamental setup and formulation of our sampled-data filter design framework in Section 4, and Section 5 gives a solution method via fast-sample/fast-hold approximation. Finally, we give some examples in signal reconstruction in Section 6.

In closing this Introduction, we note that this paper draws heavily from our very recent paper [42]. We refer the interested reader to this paper for detailed technical proofs, comparisons to other approaches, and more examples.

## 2. PRELIMINARIES

**2.1.  $H^\infty$  optimization.** Let us introduce some basic function spaces and performance measures. Let  $L^2(a, b)$  (or  $L^2[a, b]$ ,  $L^2[a, b]$ , etc.) be the space of Lebesgue square integrable functions on the interval  $(a, b)$ ,  $a < b$ . For a function  $f$  valued in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , its  $L^2$ -norm is denoted by

$$\|f\|_2 = \left\{ \int_a^b |f(t)|^2 dt \right\}^{1/2}, \quad (1)$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{C}^n$ . Let  $H^2$  denote the space of  $\mathbb{C}^n$ -valued functions  $f$  that are analytic on the open right half plane  $\mathbb{C}_+ := \{s : \text{Res} > 0\}$  and satisfy

$$\sup_{x>0} \int_{-\infty}^{\infty} |f(x + jy)|^2 dy < \infty.$$

The  $H^2$ -norm of a function  $f \in H^2$  is defined by

$$\|f\|_2 := \sup_{x>0} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + jy)|^2 dy \right\}^{1/2}. \quad (2)$$

It is well known that Laplace transform gives an isometry between  $L^2[0, \infty)$  and  $H^2$ .

The space  $H^\infty$  denotes the Hardy space of functions analytic on  $\mathbb{C}_+$  and bounded there. It is a Banach space with norm

$$\|f\|_\infty := \sup_{s \in \mathbb{C}_+} |f(s)|. \quad (3)$$

An element  $f$  of  $H^\infty$  admits nontangential limit to the imaginary axis almost everywhere, which we denote by  $f(j\omega)$ ,  $\omega \in \mathbb{R}$ . Then the  $H^\infty$ -norm of  $f \in H^\infty$  is equal to

$$\|f\|_\infty = \text{esssup}_{-\infty < \omega < \infty} |f(j\omega)|. \quad (4)$$

Now let  $G$  be the transfer function of a finite-dimensional, asymptotically stable linear continuous-time system. Then  $G$  belongs to  $H^\infty$ , and its “size” is measured by the  $H^\infty$ -norm, i.e., the supremum (or maximum) of the Bode magnitude plot as in (4).

The steady-state response of  $G$  against a sinusoid  $e^{j\omega t}$  is given by  $G(j\omega)e^{j\omega t}$ , and its magnitude is bounded as

$$|G(j\omega)e^{j\omega t}| \leq \sup_{-\infty < \omega < \infty} |G(j\omega)| \cdot |e^{j\omega t}| = \|G\|_\infty.$$

In general, for  $u \in H^2$ , it is known that

$$\|Gu\|_2 \leq \|G\|_\infty \cdot \|u\|_2, \quad (5)$$

and this bound is tight. Hence the  $H^\infty$  norm gives the  $L^2$  energy induced-gain, and minimizing it yields a system that works uniformly well for the whole frequency range<sup>1</sup>.

For this reason, it is recognized that the  $H^\infty$ -norm criterion is often superior to the  $H^2$ -norm criterion, where the  $H^2$ -norm for a stable matrix transfer function is defined as

$$\|G\|_2 := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \{G^*(j\omega)G(j\omega)\} dt \right)^{1/2}.$$

The  $H^\infty$  norm has been used successfully in the control literature [7, 8, 11].

<sup>1</sup>However, it is to be noted that it is not possible to uniformly attenuate  $|G(j\omega)|$ . If we attenuate  $G(j\omega)$  for a certain frequency range, it will yield an amplification at another range. Due to this effect, one usually introduce a frequency weighting  $W(s)$ , and minimize  $|W(j\omega)G(j\omega)|$ .

**2.2. Lifting, transfer functions, and frequency responses.** The  $H^\infty$  norm criterion is naturally extended to sampled-data systems. The problem here is that such systems have two time sets: continuous and discrete. Hence the overall system is not time-invariant in the classical sense. This difficulty can be remedied by the now-standard technique called lifting [3, 4, 33, 34], which converts a linear time-invariant continuous-time system to an infinite-dimensional discrete-time system. It is then possible to naturally extend the notion of the  $H^\infty$ -norm to sampled-data systems. To be more precise, let  $\mathcal{G}$  denote the input/output operator of such a system. Then its  $H^\infty$ -norm is defined to be the induced norm against all  $L^2$  inputs:

$$\|\mathcal{G}\|_\infty := \sup_{u \in L^2, u \neq 0} \frac{\|\mathcal{G}u\|_2}{\|u\|_2}. \quad (6)$$

Via lifting, this norm is indeed equivalent to the maximum gain of the frequency response operator of  $\mathcal{G}$  as in (3).

We start by placing a continuous-time signal in a discrete-time framework. Take a continuous-time signal  $w(t)$ , and consider the following mapping  $\mathcal{L}$  (with a suitable domain and codomain) that maps  $w$  into a sequence of functions as

$$(\mathcal{L}w)[k] := \mathbf{w}[k] := \{w(kh + \theta)\}_{\theta \in [0, h)}, k = 0, 1, 2, \dots \quad (7)$$

See Fig. 1. The operator  $\mathcal{L}$  is called lifting.

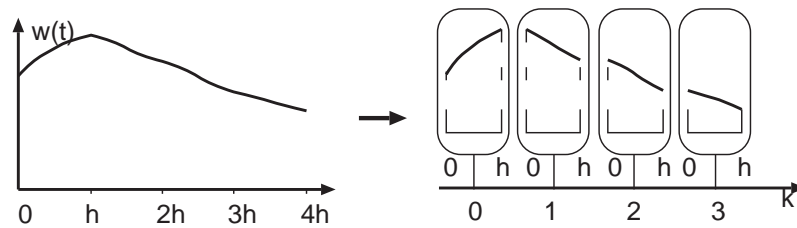


Figure 1. Lifting: a continuous-time signal  $w(t)$  (left) is converted to a function-valued discrete time signal (right)

This idea makes it possible to view time-invariant, or even periodically time-varying continuous-time systems as linear, time-invariant discrete-time systems.

Using this operator, one can describe a linear, time-invariant continuous-time system with a linear, time-invariant discrete-time system. Consider the following linear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad (8)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  and  $y(t) \in \mathbb{R}^p$  are the state, input and output of this system, respectively. Let us assume, e.g.,  $u \in L^2_{\text{loc}}[0, \infty)$ , the set of locally square-integrable functions on  $[0, \infty)$ . The idea is that we view the continuous-time system (8) as one with discrete-timing  $t = kh$ ,  $k = 0, 1, 2, \dots$  such that it receives function-valued inputs at these instants and produces function-valued outputs at these times also. Suppose that (8) is at state  $x(kh)$  at time  $t = kh$ . Then

$$\begin{aligned} x(kh + h) &= e^{Ah}x(kh) + \int_0^h e^{A(h-\tau)}Bu(kh + \tau)d\tau, \\ y(kh + \theta) &= Ce^{A\theta}x(kh) + \int_0^\theta e^{A(\theta-\tau)}Bu(kh + \tau)d\tau. \end{aligned}$$

where  $0 \leq \theta < h$  denotes the intersample parameter. Lifting the input  $u(t)$  and the output  $y(t)$  as per (7), we can rewrite these formulas as a lifted discrete-time system [4, 34]:

$$\begin{aligned} x[k+1] &= \mathcal{A}x[k] + \mathcal{B}u[k], \\ \mathbf{y}[k] &= \mathcal{C}x[k] + \mathcal{D}u[k], \quad k = 0, 1, 2, \dots, \end{aligned}$$

where  $x[k] = x(kh)$ ,  $\mathbf{u}[k] = (\mathcal{L}u)[k]$ ,  $\mathbf{y}[k] = (\mathcal{L}y)[k]$ , and

$$\begin{aligned} \mathcal{A} : \mathbb{R}^n &\rightarrow \mathbb{R}^n : x \mapsto e^{Ah}x, \\ \mathcal{B} : L^2[0, h] &\rightarrow \mathbb{R}^n : u \mapsto \int_0^h e^{A(h-\tau)}Bu(\tau)d\tau \\ \mathcal{C} : \mathbb{R}^n &\rightarrow L^2[0, h] : x \mapsto Ce^{A\theta}x, \\ \mathcal{D} : L^2[0, h] &\rightarrow L^2[0, h] : u \mapsto \int_0^\theta Ce^{A(\theta-\tau)}Bu(\tau)d\tau, \end{aligned} \tag{9}$$

where  $\theta \in [0, h)$  describes the intersample parameter. Observe that the operators  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  above do not depend on time  $k$ , and hence system (9) is a time-invariant discrete-time system, albeit with infinite-dimensional input and output spaces. Hence it is straightforward to connect this system with a discrete-time controller (or a filter), and the obtained sampled-data system is again a linear, time-invariant discrete-time system without sacrificing any intersampling information. The resulting system can also be described by a 4-tuple of operators  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ , and its transfer function (operator) of the lifted system is defined as

$$\mathcal{G}(z) = \mathcal{D} + \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}$$

with such  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ . Note that for each fixed  $z \in \mathbb{C} \setminus \sigma(e^{Ah})$ , ( $\sigma(e^{Ah}) :=$  the set of eigenvalues of  $e^{Ah}$ ),  $\mathcal{G}(z)$  is a linear operator acting on  $L^2[0, h]$  into itself. The frequency response operator is then defined as  $\mathcal{G}(e^{j\omega h})$ , and the gain at frequency  $\omega$  is defined as

$$\|\mathcal{G}(e^{j\omega h})\| = \sup_{\substack{v \in L^2[0, h] \\ v \neq 0}} \frac{\|\mathcal{G}(e^{j\omega h})v\|}{\|v\|}.$$

The  $H^\infty$  norm of  $\mathcal{G}$  then becomes

$$\|\mathcal{G}\|_\infty = \sup_{\omega \in [0, 2\pi/h)} \|\mathcal{G}(e^{j\omega h})\|,$$

which is known to be identical to the  $L^2$ -induced norm given by (6) [4].

### 3. SIGNAL RECONSTRUCTION AND SAMPLING THEOREM

Consider the block diagram depicted in Fig. 2.

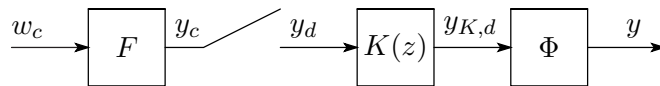


Figure 2. Signal Reconstruction System

In this diagram, the signal  $w_c \in L^2$  denotes the external analog signal to be reconstructed. It is filtered by an analog filter (acquisition device)  $F$ , and then sampled by the sampler with sampling period  $h$ . If  $f(t)$  denotes the impulse response of the analog filter  $F$ , then the discrete-time signal  $y_d[k]$  is easily seen to be given by

$$y_d[k] = (f * w_c)[k] = \langle \check{f}(\cdot - kh)w_c \rangle, \quad (10)$$

where  $\check{f}(t) := f(-t)$  is the mirror image (with respect to time) of  $f$ , and  $\langle f, g \rangle$  denotes the inner product in  $L^2$ . The obtained signal  $y_d$  is then processed by a discrete-time filter  $K$  and then the filtered discrete-time signal  $y_{K,d}$  is converted back to an analog signal  $y$  via a reconstruction device  $\Phi$ . Denoting by  $\phi$  the impulse response of  $\Phi$ , the reconstructed  $y$  is given by

$$y(t) = \sum_{k=-\infty}^{\infty} y_{K,d}[k]\phi(t - kh). \quad (11)$$

In the Shannon paradigm, the analog filter  $F$  is taken to be the ideal filter, and  $\phi$  above is the sinc function [44, 30]. As mentioned in the Introduction, this has several limitations. To take care of this, one often employs an approximation of the ideal filter with respect to  $H^2$  norm [10], and this unfortunately yields a sharp ringing effect in the frequency domain.

Unser and co-workers published series of papers of generalized sampling theorems where the acquisition device  $F$  is not the ideal filter [28, 29, 30]. First define the subspace

$$V_f := \left\{ \sum_{k=-\infty}^{\infty} \alpha[k]f(t - kh) : \{\alpha[k]\} \in \ell^2 \right\} \quad (12)$$

generated by the translates of the impulse response of the acquisition filter, and the reconstruction space

$$V_\phi := \left\{ \sum_{k=-\infty}^{\infty} \beta[k]\phi(t - kh) : \{\beta[k]\} \in \ell^2 \right\} \quad (13)$$

generated by the translates of the reconstruction function  $\phi$ . From the consistency requirement [28], a key step in their procedure is the oblique projection of  $L^2$  onto  $V_\phi$  perpendicular to  $V_f$ . A precise comparison of this approach with our work is given in [42].

#### 4. $H^\infty$ SIGNAL RECONSTRUCTION PROBLEM

We are now ready to precisely state our signal reconstruction problem. The basic features are the following:

- We allow a finite step preview for reconstruction.
- The acquisition device, sampling and hold elements are fixed.

Consider the block diagram Fig. 3.

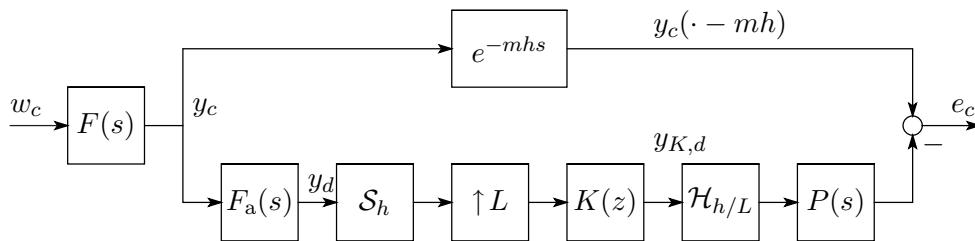


Figure 3. Error system of a sampled-data design filter

The external continuous-time signal  $w_c \in L^2$  is first filtered, or band-limited (mildly but not perfectly) by going through the analog low-pass filter  $F(s)$ , which is linear and time-invariant, and finite-dimensional. This  $F(s)$  is a rational function of  $s$  which is strictly proper (i.e., the degree of the numerator polynomial is less than that of the denominator). As is well known, it is represented by a linear, time-invariant system

$$\begin{aligned}\frac{dx}{dt}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}$$

where  $A, B, C$  are constant matrices of appropriate sizes, and  $F(s) = C(sI - A)^{-1}B$ . Hence  $F$  is not an ideal filter unlike the case of the Shannon paradigm, and is physically realizable through the above state space model. The signal  $w_c$  is the external signal that drives  $F$  and produces the actual signal  $y_c$  to be processed. That is, we assume that the original analog signals to be sampled are in the following subspace of  $L^2$ :

$$FL^2 := \{y_c \in L^2 : y_c = Fw_c, w_c \in L^2\}.$$

It is proved in [24] that the band-limited signal subspace

$$BL := \{y_c \in L^2 : \text{supp } \hat{y}_c \subset (-\pi/h, \pi/h)\},$$

is a proper subset of  $FL^2$ , that is,  $BL \subsetneq FL^2$ . The filter  $F(s)$  is chosen based on the following guidelines:

- a frequency distribution of input analog signals obtained by averaging or enclosing gains of their Fourier transforms.
- a dynamical model of signal generator such as musical instruments.

The example in Section 6 gives a brief guideline on how to choose  $F(s)$  based on the envelope of energy distributions of the signal. Note that when  $F$  is ideal, then the class we are dealing with agrees with the ideal sampling theorem.

The produced signal  $y_c$  is then sampled by ideal sampler  $\mathcal{S}_h$ , filtered by an anti-aliasing filter  $F_a(s)$ , and becomes a discrete-time signal  $y_d$  with sampling period  $h$ . This signal is then upsampled by  $\uparrow L$  to allow for processing (interpolation) between the original sampling period  $h$ . The digital filter  $K(z)$  processes this upsampled signal to produce  $y_{K,d}$ . The signal  $y_{K,d}$  then goes through the zero-order hold  $\mathcal{H}_{h/L}$  and becomes a continuous-time signal. It is then further processed by an analog low-pass filter  $P(s)$  to become the final analog output  $y_p$ .

In the upper part of the diagram, we allow  $m$  steps of delay for the analog signal  $y_c$  and obtain  $y_c(t - mh)$ . This is a setup for allowing a “preview” of  $y_c$  for  $m$  samples by the proper filter transfer function  $K(z)$ . It is very effective compared to reconstruction without a preview. This also takes care of certain processing delays caused by the processing filter. The integer  $m$  is a design parameter that can be chosen by the designer. This is in marked contrast to the conventional design methodologies: These methods usually allow a non-causal impulse response for reconstruction, e.g., [32, 21, 22]. But in real implementation, one has to truncate it, and it is often unclear how many steps one would need to obtain a desired accuracy. In the present setup, one can prespecify an allowable step of delays (preview), and obtain an optimal design under such a constraint.

Finally, the processed signal  $y_p$  is compared with this delayed  $y_c(t - mh)$  and subtracted from it to obtain the error signal  $e_c$ . The design objective is to make the error as small as possible. Observe also that this design framework is formulated in the continuous-time domain in contrast to the usual discrete-time setups.

We must specify a performance index to give a precise meaning to this problem. The following  $L^2$  induced norm from  $w_c$  to  $e_c$  (or the sampled-data  $H^\infty$  norm) is the one we take:

$$J := \sup_{w_c \in L^2, w_c \neq 0} \frac{\|e_c\|_2}{\|w_c\|_2}. \quad (14)$$

We thus arrive at the following design problem:



**Problem 1.** Let  $T_{ew}$  denote the input/output operator from  $w_c$  to  $e_c(\cdot) := y_c(\cdot) - u_c(\cdot - mh)$  in Fig. 3. Given an attenuation level  $\gamma > 0$ , find, if one exists, a stable digital (discrete-time) filter  $K(z)$  such that

$$J := \|T_{ew}\|_\infty = \sup_{w_c \in L^2[0, \infty)} \frac{\|T_{ew}w_c\|_2}{\|w_c\|_2} < \gamma. \quad (15)$$

The performance index (15) intends to minimize the maximum error induced by an (unknown) input  $w_c$  that gives rise to the largest norm of  $e_c$  among all inputs. This is made possible by the  $H^\infty$  design methodology. Note that the actual error is not known, but due to the min-max nature of the problem, we can minimize the worst transmission error. Observe also that this setup allows for a capability of minimizing continuous-time phase errors due to the continuous-time nature of the performance index, as opposed to the conventional gain-phase design principles.

This min-max problem differs sharply from the orthogonal projection based methods. Also, due to sampling,  $T_{ew}$  is not even time-invariant (in continuous-time).

It is now known however that this problem is reducible to a linear time-invariant problem via lifting; see Subsection 2.2; the problem is now solvable via now-standard  $H^\infty$  control theory, see, e.g., [4, 3] (see also [8] for standard treatments of  $H^\infty$  control in the continuous-time setting).

The existence of  $e^{-mhs}$  makes this an infinite-dimensional  $H^\infty$  problem; see [19, 35, 39, 41], etc. The simplest solution is to employ the so-called fast-sampling/fast-hold approximation, which we will outline in the next section.

## 5. SOLUTION VIA FAST-SAMPLING/FAST-HOLD APPROXIMATION

While Problem 1 is known to be reducible to a finite-dimensional problem [19, 23], it is not necessarily appealing computationally. It is often more convenient to resort to an approximation method. We employ the fast sample/hold approximation [18, 4, 35, 39]. This method approximates continuous-time inputs and outputs via a sampler and hold that operate in the period  $h/N$  for some positive integer  $N$ . The method usually works fairly well for  $N \sim 5L$ , where  $L$  is the upsampling ratio given in Section 4, and the convergence of such an approximation is shown in [35, 39]. We show here the design procedure of  $K(z)$  by the fast sampling/hold approximation.

The error system in Fig. 3 is a multirate system due to the upsampler  $\uparrow L$ . We first reduce this system to a single-rate one. Introduce the *discrete-time lifting*, also known as the *polyphase decomposition* [43],  $\mathbf{L}_L$  and its inverse  $\mathbf{L}_L^{-1}$  as

$$\begin{aligned} \mathbf{L}_L &:= (\downarrow L) \begin{bmatrix} 1 & z & \dots & z^{L-1} \end{bmatrix}^T, \\ \mathbf{L}_L^{-1} &:= \begin{bmatrix} 1 & z^{-1} & \dots & z^{-L+1} \end{bmatrix} (\uparrow L). \end{aligned} \quad (16)$$

Then  $K(z)(\uparrow L)$  can be rewritten by a lifted system as

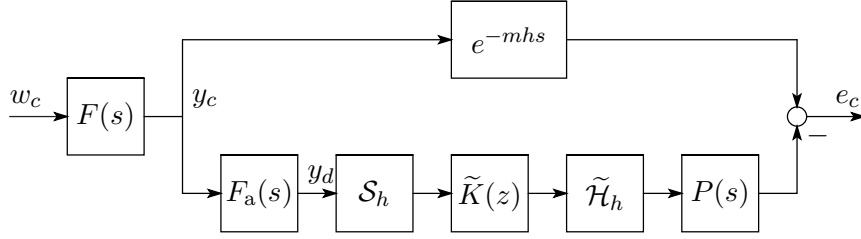
$$\begin{aligned} K(z)(\uparrow L) &= \mathbf{L}_L^{-1} \tilde{K}(z), \\ \tilde{K}(z) &:= \mathbf{L}_L K(z) \mathbf{L}_L^{-1} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T. \end{aligned}$$

The filter  $\tilde{K}(z)$  is an LTI (linear and time-invariant), single-input/ $L$ -output system. Define  $\tilde{\mathcal{H}}_h := \mathcal{H}_{h/L} \mathbf{L}_L^{-1}$ , and we obtain the following equality

$$\mathcal{H}_{h/L} K(z)(\uparrow L) \mathcal{S}_h = \tilde{\mathcal{H}}_h \tilde{K}(z) \mathcal{S}_h.$$

Hence the multirate system in Fig. 3 is reduced to the single-rate system shown in Fig. 4.



Figure 4. Reduced single-rate error system  $T_{ew}$ 

We then employ the fast sample/hold approximation for the error system  $T_{ew}$  in Fig. 4. We connect fast sample and hold devices  $\mathcal{S}_{h/N}$ ,  $\mathcal{H}_{h/N}$ , and the discrete-time lifting  $\mathbf{L}_N$  with the error system  $T_{ew}$  as shown in Fig. 5.

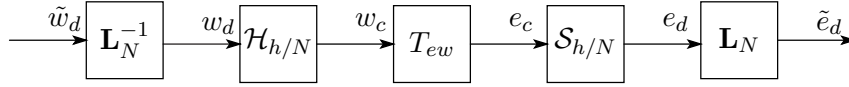


Figure 5. Fast-sampling/Fast-hold (FSFH) transformation

We call this fast-sampling/fast-hold (FSFH) transformation and denote it by  $\mathbf{FSFH}(T_{ew}, h, N)$ .

Before we give the design formula of  $\mathbf{FSFH}(T_{ew}, h, N)$ , we introduce FSFH transformation for continuous-time LTI systems. For brevity of notation, let us adopt the following shorthand notation for continuous-time transfer function  $D + C(sI - A)^{-1}B$  or discrete-time transfer function  $D + C(zI - A)^{-1}B$ :

$$\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \begin{cases} D + C(sI - A)^{-1}B, & \text{for continuous-time systems,} \\ D + C(zI - A)^{-1}B, & \text{for discrete-time systems.} \end{cases}$$

Let  $\mathbf{c2d}$  denote the step-invariant transformation [4], that is,

$$\mathbf{c2d} \left( \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], h \right) := \mathcal{S}_h \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \mathcal{H}_h = \left[ \begin{array}{c|c} e^{Ah} & \int_0^h e^{A(h-t)} B dt \\ \hline C & D \end{array} \right].$$

Also, let  $\mathbf{lift}$  denote the discrete-time lifting transformation [4], that is,

$$\mathbf{lift} \left( \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], N \right) := \mathbf{L}_N \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \mathbf{L}_N^{-1} = \left[ \begin{array}{c|ccc} A^N & A^{N-1}B & A^{N-2}B & \dots & B \\ \hline C & D & 0 & \dots & 0 \\ CA & CB & D & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ CA^{N-1} & CA^{N-2}B & CA^{N-3}B & \dots & D \end{array} \right].$$

Then, for continuous-time LTI system  $F$ , the FSFH transformation of  $F$  is given by

$$\mathbf{FSFH}(F, h, N) := \mathbf{lift}(\mathbf{c2d}(F, h/N), N).$$

By the FSFH transformation, the sampled-data error system  $T_{ew}$  can be approximated by a discrete-time LTI system as in the following theorem:

**Theorem 1.** Let  $N = ll$ , where  $l$  is a positive integer, and define the discrete-time LTI system  $T_N$  as follows:

$$T_N(z) = z^{-m} F_N(z) - P_N(z) H \tilde{K}(z) S \tilde{F}_N(z), \quad (17)$$

where

$$F_N := \mathbf{FSFH}(F, h, N), \quad P_N := \mathbf{FSFH}(P, h, N), \quad \tilde{F}_N := \mathbf{FSFH}(FF_a, h, N),$$

$$H := \text{diag}\{I_l\} \in \mathbb{R}^{N \times L}, \quad I_l := [1, 1, \dots, 1]^T \in \mathbb{R}^l, \quad S := [1, 0, \dots, 0] \in \mathbb{R}^{1 \times N}.$$

Then, for each fixed  $\tilde{K}$  and for each  $\omega \in [0, 2\pi/h)$ , the frequency response

$$\|T_N(e^{j\omega h})\| \rightarrow \|T_{ew}(e^{j\omega h})\|, \quad (18)$$

as  $N \rightarrow \infty$ , and this convergence is uniform with respect to  $\omega \in [0, 2\pi/h)$ . Furthermore, this convergence is also uniform in  $\tilde{K}$  if  $\tilde{K}$  ranges over a compact set of filters.

The proof is almost the same as in [42, Theorem 1].

In view of the uniformity of convergence  $\|T_N\|_\infty$  in  $\tilde{K}$ , our design problem (15) can be approximated by

$$\|T_N\|_\infty < \gamma.$$

This is a discrete-time  $H^\infty$  optimization problem. To obtain a filter  $\tilde{K}(z)$  satisfying the above inequality, we can adopt numerical softwares as MATLAB with robust control toolbox [20], by the generalized plant representation depicted in Fig. 6, where  $\tilde{w}_d = \mathbf{L}_N w_d$  and  $\tilde{e}_d = \mathbf{L}_N e_d$ .

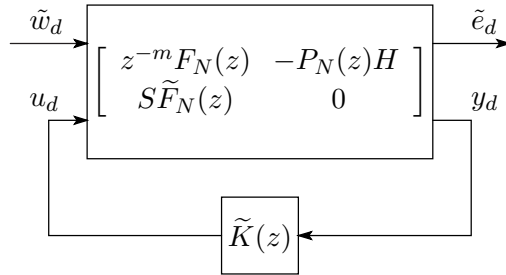


Figure 6. Discrete-time system for  $H^\infty$  optimization

Once the optimal filter  $\tilde{K}(z)$  is obtained, one can obtain the interpolation filter  $K(z)$  by the following formula:

$$K(z) = [1 \quad z^{-1} \quad \dots \quad z^{-L+1}] \tilde{K}(z^L).$$

## 6. DESIGN EXAMPLES

Let us make a comparison with a usual linear phase filter—the equi-ripple FIR filter obtained by Parks-McClellan method [43, 45] with 64 taps. Parks-McClellan method is widely used for designing FIR filters for interpolation. We design the proposed filter  $K(z)$  with interpolation ratio  $L = 4$ , sampling period  $h = 1$ , and delay step  $m = 4$ . The analog filters  $F(s)$ ,  $F_a(s)$ , and  $P(s)$  are given by

$$F(s) = \frac{1}{(Ts + 1)(0.1Ts + 1)}, \quad T = 7.0187,$$

$$F_a(s) = \frac{1}{0.01s + 1}, \quad P(s) = \frac{1}{0.05s + 1}.$$

Reflecting a typical energy distribution of orchestral music, the time constant  $T = 7.0187$  is taken to be equivalent to 1 kHz with sampling frequency 44.1 kHz. It therefore corresponds to an energy distribution that decays by  $-20$  dB per decade from 1 kHz and  $-40$  dB per decade from 10 kHz.

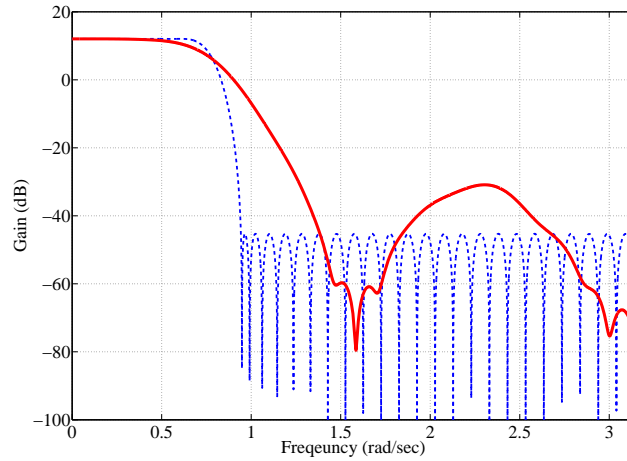


Figure 7. Bode gain plot of the proposed filter (solid) and the equi-ripple filter (dash)

Fig. 7 shows the Bode gain plots of the proposed filter and the equi-ripple FIR filter with 64 taps. We can see that the equi-ripple filter has a sharp decay around the cutoff frequency  $\omega = \pi/4$ , while the filter obtained by the proposed method shows a rather mild decay.

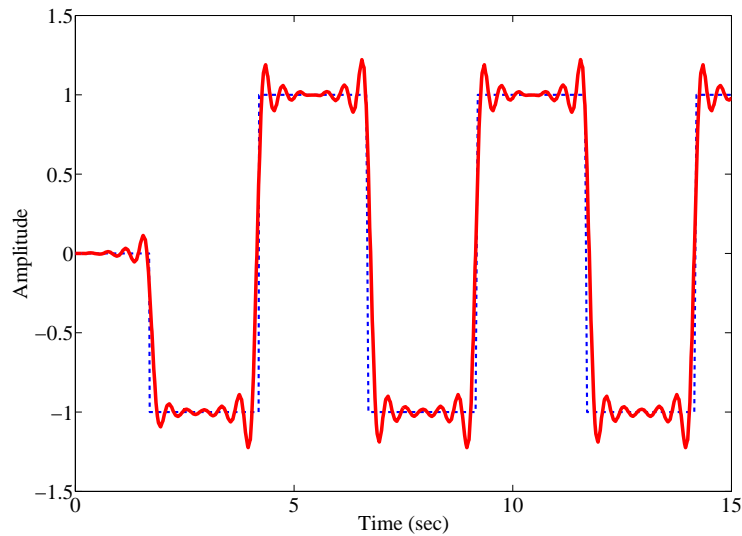


Figure 8. Response of the equi-ripple filter against a rectangular wave

Fig. 8 shows the response of the equi-ripple filter against a rectangular wave. It exhibits a very sharp ringing effect. This is because the filter has a sharp cut-off characteristic, and inevitably introduces the well-known Gibbs phenomenon due to the fact that the frequency components beyond the pass-band are sharply truncated. In contrast, Fig. 9 shows the response of the filter designed by the present method. It shows virtually no ringing.

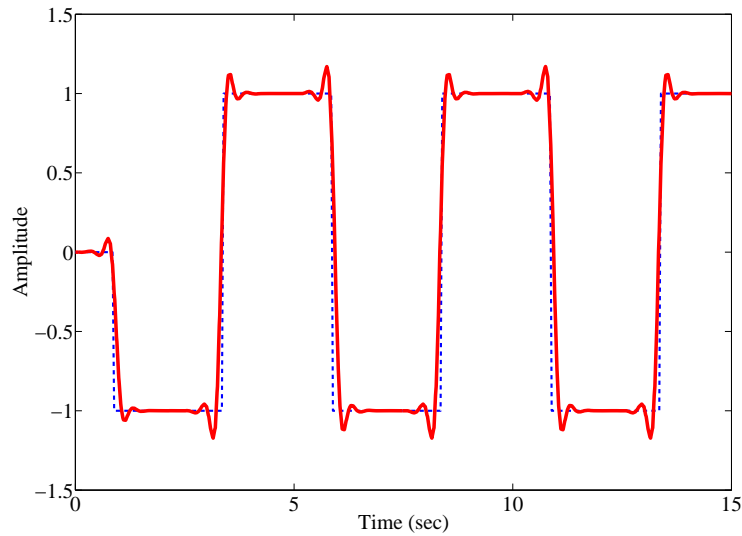


Figure 9. Response of a sampled-data design filter against a rectangular wave

Fig. 10 shows the frequency response gain of the sampled-data error system  $T_{ew}$ . We design filters by the proposed method for reconstruction delay  $m = 1, 2, 3, 4, \dots$ , and compare them with the 64-tap equi-ripple filter. For all  $m$ , the  $H^\infty$  norm of  $T_{ew}$  does not change, but the gain reduces for almost all frequencies as  $m$  increases. For  $m \geq 4$ , the frequency response gain remains unchanged. This indicates that  $m = 4$  is optimal from the point of view that smaller delay is better for signal processing. The equi-ripple filter exhibits large errors in the whole frequency range as compared with the proposed filter when  $m \geq 4$ . Note that the equi-ripple filter has the reconstruction delay  $m = 16$ . These errors give an explanation of the ringing effect in Fig. 8.

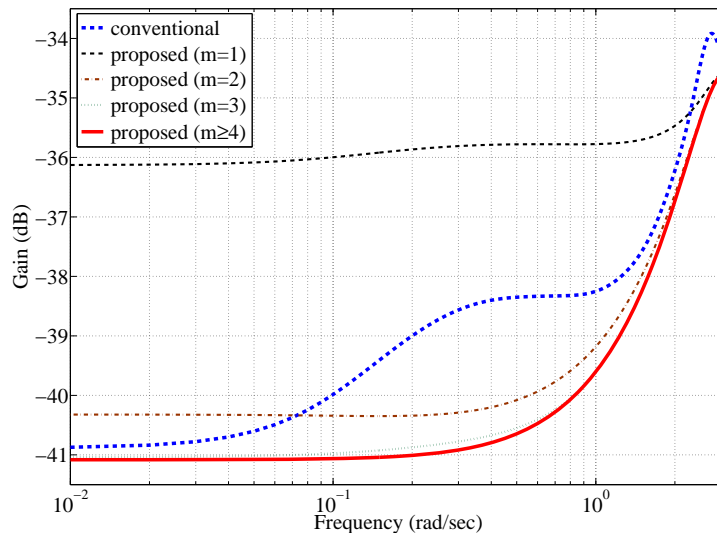


Figure 10. Frequency response of error system  $T_{ew}$  with sampled-data designed filter (solid) and the equi-ripple filter (dash)

Fig. 11 shows the relation between the upsampling ratio  $L$  and the achieved performance  $\|T_{ew}\|_\infty$ . As illustrated in this figure, the performance improves as the upsampling ratio  $L$

increases, but for  $L \geq 4$ , the performance is nearly unchanged. This figure suggests that  $L = 4$  is nearly optimal since for large  $L$  leads to high sampling frequency  $L\pi$  (rad/sec).

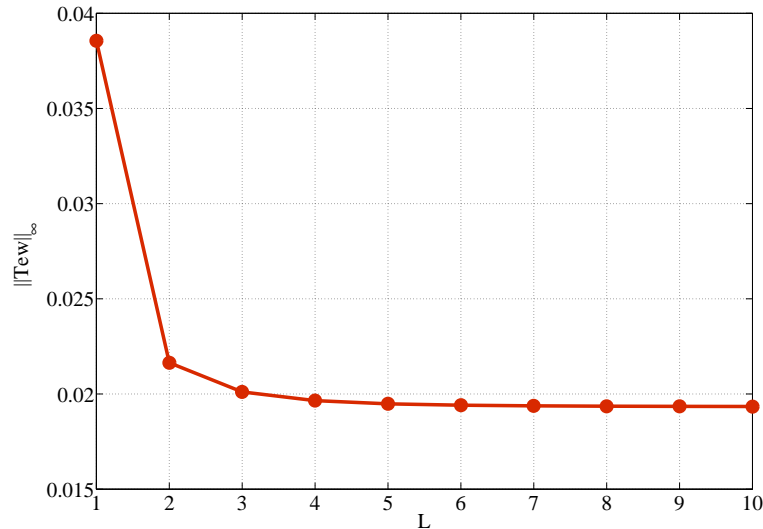


Figure 11. Upsampling ratio  $L$  versus performance  $\|T_{ew}\|_{\infty}$

## 7. CONCLUDING REMARKS

We have presented an overview on a new framework for digital signal processing. The fundamental philosophy is the emphasis on analog (continuous-time) performance with discrete-time signal processing. This naturally leads to a technical difficulty because of the two different time-sets involved: continuous and discrete. Leveraging sampled-data  $H^{\infty}$  control theory, we have presented computable procedures for designing optimal, stable, causal filters. These filters are optimal with respect to a uniform analog performance measure. Our methodology is applicable to a wide variety of theoretical and application problems in digital signal processing. We hope that this article helps the readers familiarize themselves to this new approach and that it will be more widely used in the future.

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